Seminar at LAAS

Numerical Computational Techniques for Nonlinear Optimal Control

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* Joint work with N. Sakamoto and T. Nakamura

1. Introduction

Optimal control of a nonlinear system

$\cdots$ Hamilton--Jacobi--Bellman eq.

- Power series method
  [Al'brekht 61][Lukes 69]...

- Stable-manifold method
  [Sakamoto--van der Schaft 08]
  cf. [Yamashita--Shima 98]

  swing-up of a pendulum,
  control under input saturation,
  nonlinear optimal servo, ...

This talk: Numerical computational techniques for improvement of the stable-manifold method

- Time evolution of a Hamiltonian system

  $\cdots$ numerically unstable to compute

  $\rightarrow$ numerical methods that preserve the structure of the Hamiltonian system

- Choice of initial points $\cdots$ based on trials and errors

  $\rightarrow$ shooting method

2. Stable-manifold method

Problem

Given a plant $\dot{x}(t) = f(x(t)) + G(x(t))u(t)$,

obtain an input $u(t)$ that minimizes

the objective fn. $\int_0^\infty x(t)^\mathrm{T} Q x(t) + u(t)^\mathrm{T} R u(t)\, \mathrm{d}t.$

where $f(0) = 0,\ Q \succ O,\ R \succ O$

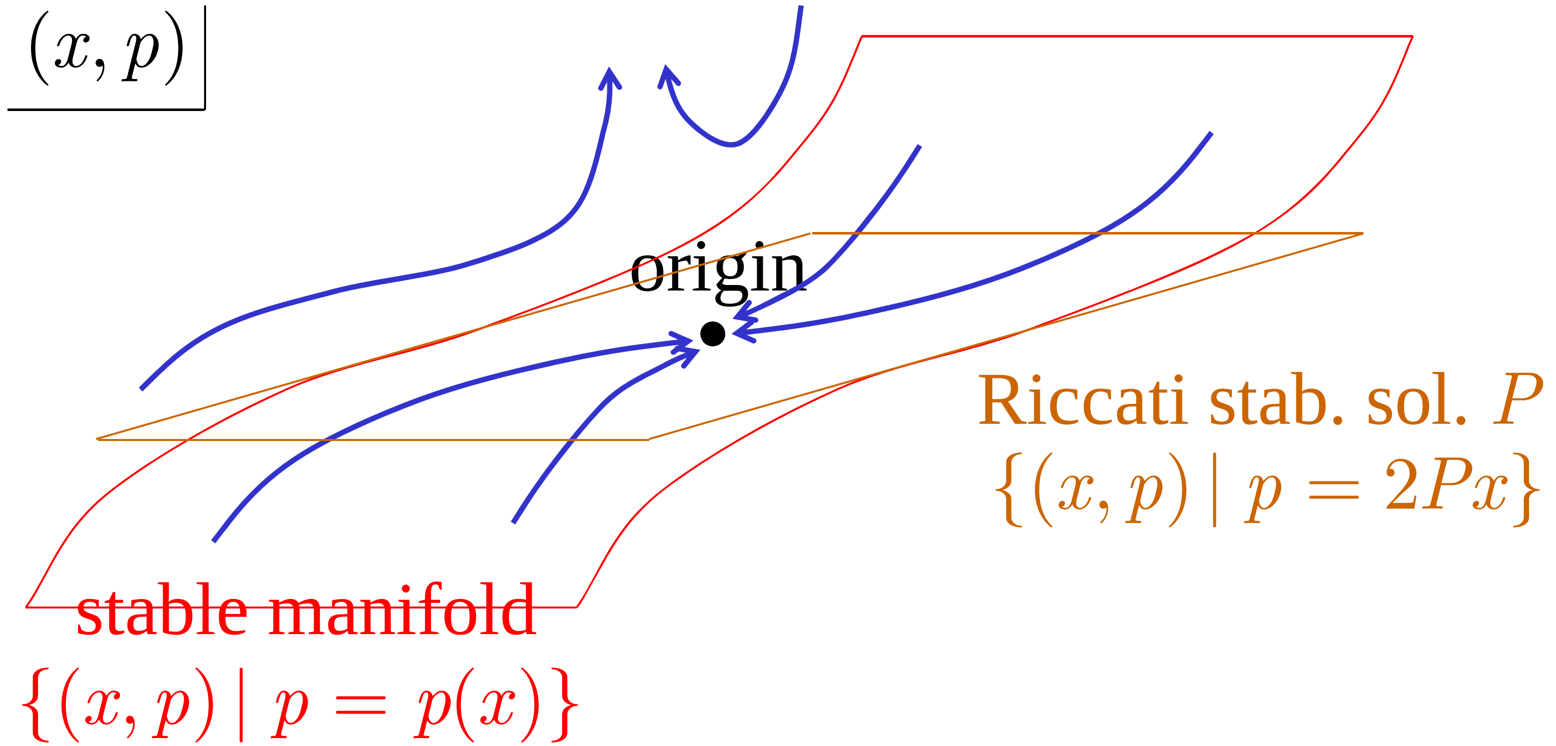
- Hamiltonian

$$H(x, p) = p^\mathrm{T} f(x) - \frac{1}{4} p^\mathrm{T} g(x) R^{-1} g(x)^\mathrm{T} p + x^\mathrm{T} Q x$$

- Hamiltonian system

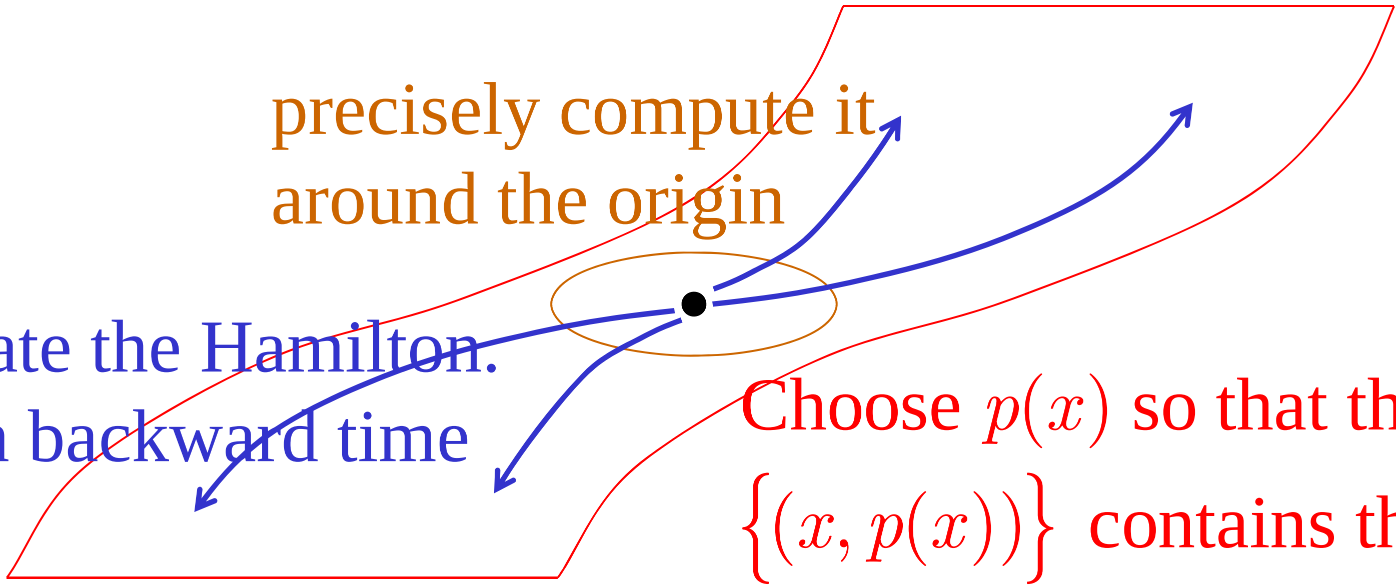
$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t))^\mathrm{T}, \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t))^\mathrm{T}$$

Hamiltonian sys. $\dot{x}(t) = \frac{\partial H}{\partial p}(x,p)^{\mathrm{T}}, \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x,p)^{\mathrm{T}}$



- Optimal input: $u(t) = -\frac{1}{2}R^{-1}g(x)^{\mathrm{T}}p(x)$

Stable-manifold method



Issues

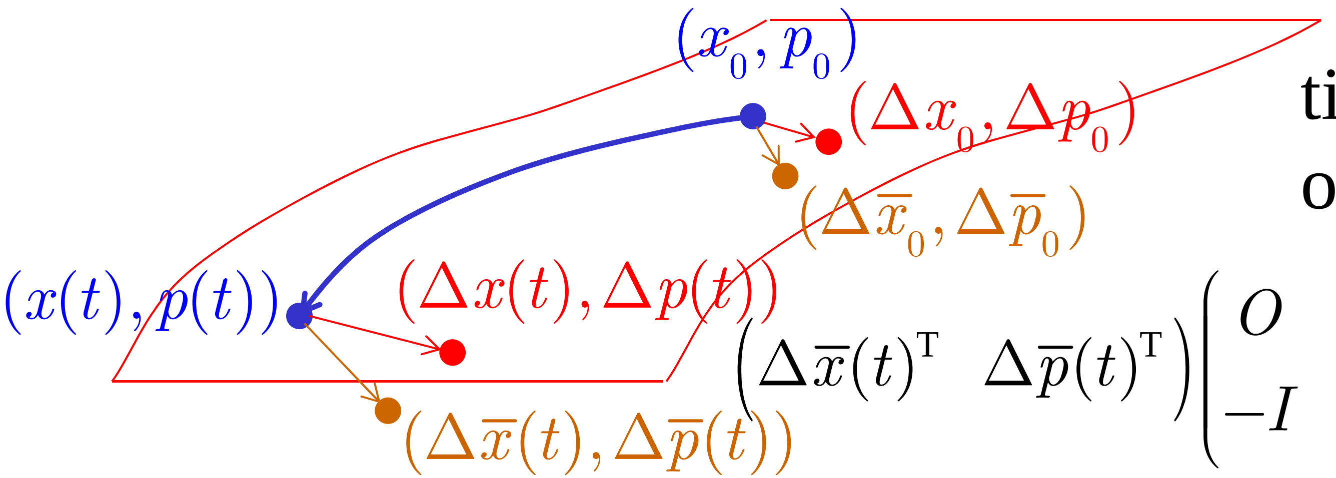
- Simulation of the Hamilton. sys. is numeric. unstable
  $\longrightarrow$ numerical methods that preserve the structure
  of the Hamiltonian system
- Choice of initial points is based on trials and errors
  $\longrightarrow$ shooting method

3. Structure preservation

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t))^{\mathrm{T}}, \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t))^{\mathrm{T}}$$

initial point: $(x(0), p(0)) = (x_0, p_0)$

- Symplecticity



time evolution
of the perturbation

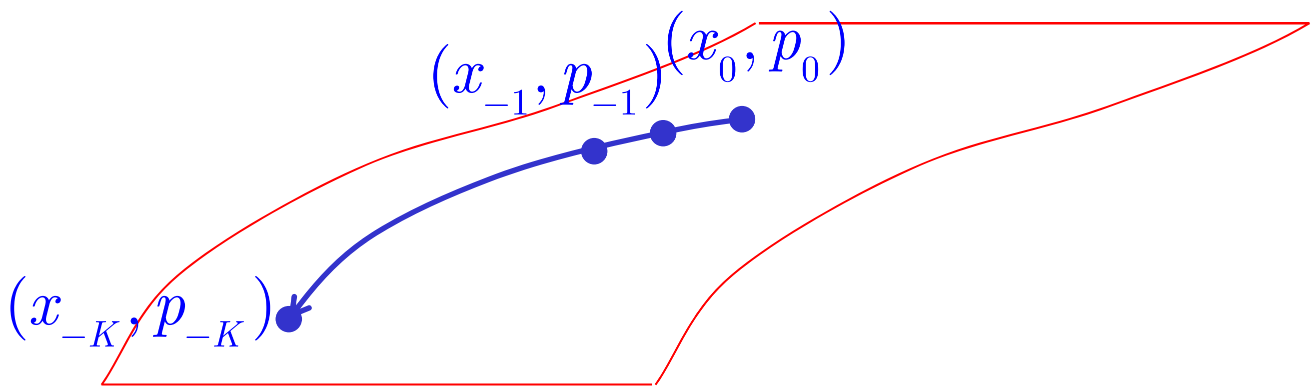
$$\begin{pmatrix} \Delta \bar{x}(t)^{\mathrm{T}} & \Delta \bar{p}(t)^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \begin{pmatrix} \Delta x(t) \\ \Delta p(t) \end{pmatrix} = \text{const.}$$

- Preservation of the Hamiltonian

$$H(x(t), p(t)) = \text{const.}$$

Numerical method

- Given a step size $h > 0$, obtain (x_{-k}, p_{-k}) that approximates $(x(-kh), p(-kh))$ for $k = 1, 2, \ldots, K$



- Standard $\cdots$ classic Runge--Kutta method
  $\to$ no symplecticity; no preserv. of the Hamiltonian

- Structure-preserving numerical methods
  $\to$ numerically stable [Hairer--Lubich--Wanner 02]

Gauss method (of order 4)

- Symplectic numerical method

For $k = 0,\ 1,\ \ldots,\ K-1$

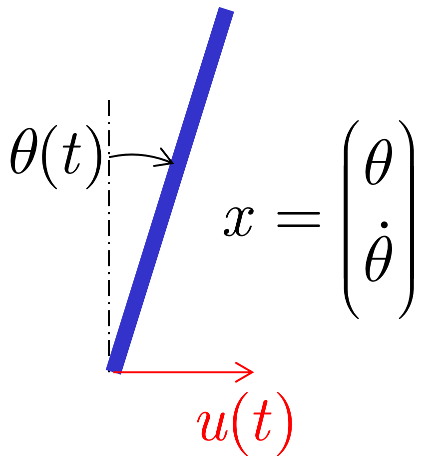
$$g_1 = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial x} \end{pmatrix}\Bigg|_{\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} x_{-k} \\ p_{-k} \end{pmatrix} + \frac{h}{4} g_1 + \left(\frac{1}{4} - \frac{\sqrt{3}}{6} \right) h g_2}$$

$$g_2 = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial x} \end{pmatrix}\Bigg|_{\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} x_{-k} \\ p_{-k} \end{pmatrix} + \left(\frac{1}{4} + \frac{\sqrt{3}}{6} \right) h g_1 + \frac{h}{4} g_2}$$

$$\begin{pmatrix} x_{-k-1} \\ p_{-k-1} \end{pmatrix} = \begin{pmatrix} x_{-k} \\ p_{-k} \end{pmatrix} + \frac{h}{2}\left(g_1 + g_2 \right)$$

- Need to solve nonlinear eqs. for g_1 and g_2

Example: swing-up of a pendulum [Sakamoto 13]

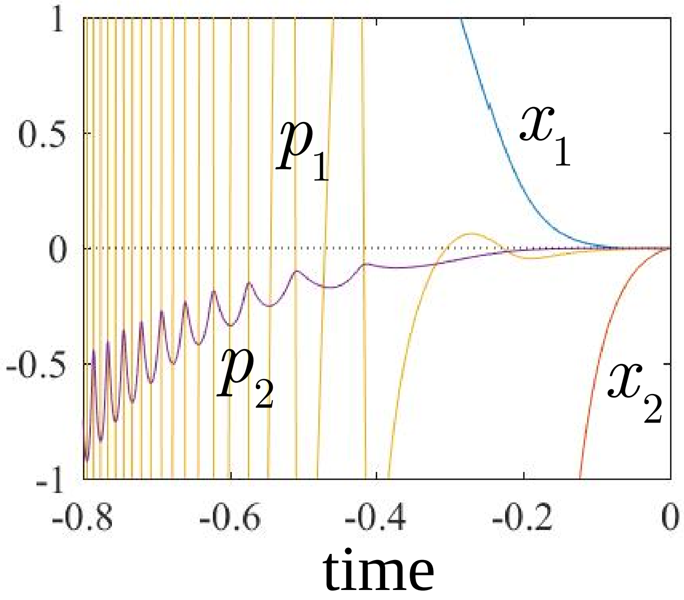


$$Q = 0.01I, \quad R = 2$$

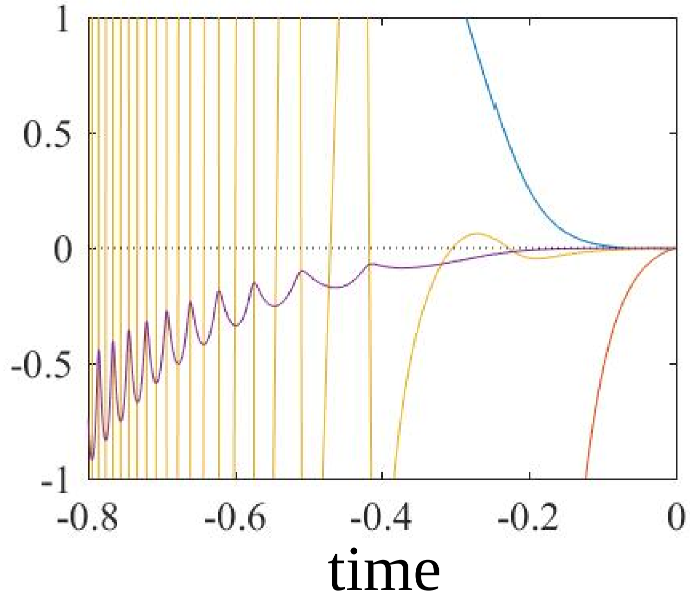
$$P = \text{Riccati stabilizing solution}$$

$$(x_0, p_0) = (x_0, 2Px_0), \quad x_0 = (-0.005 \;\; 0)^{\mathrm{T}}$$

$$h = 0.001$$



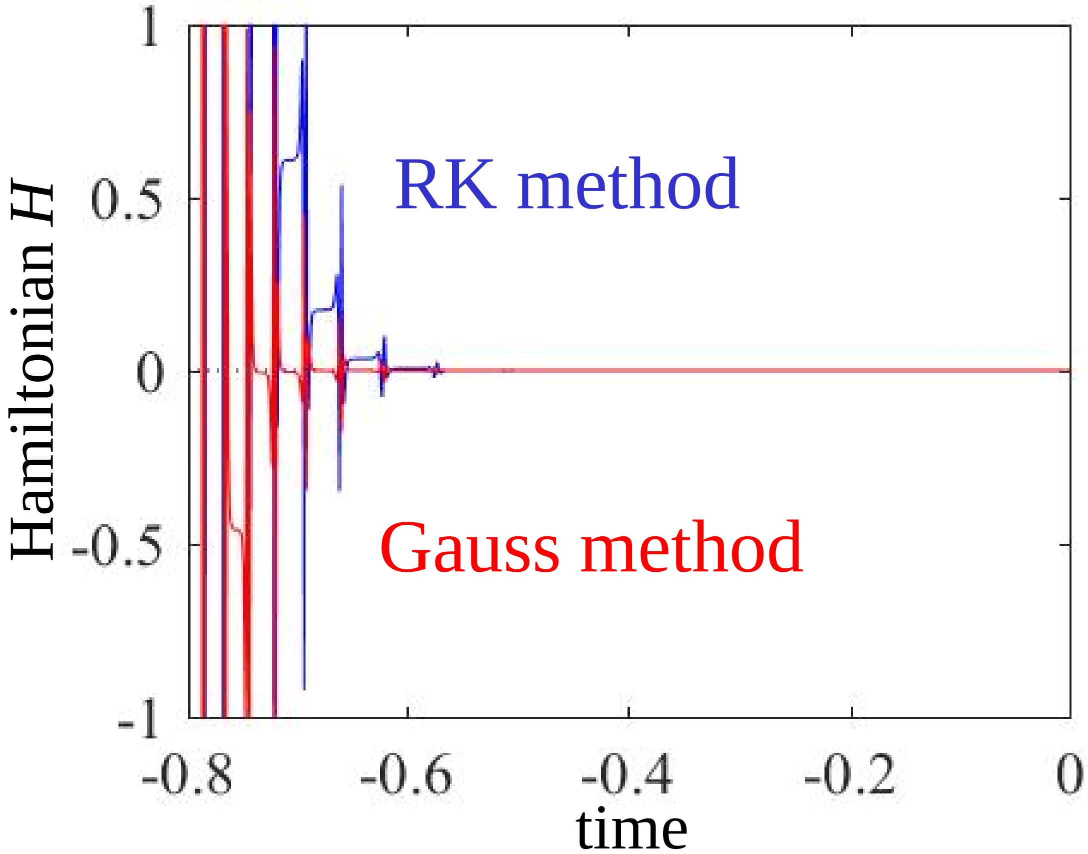
RK method



Gauss method

Values of the Hamiltonian

- $H(x, p)$ is constant theoretically
- It can be used to see the quality of computation



- Gauss method is a little better?

Averaged-vector-field method [Quispel--McLaren 08]

- Preserving the Hamiltonian value

For $k = 0,\ 1,\ \ldots,\ K-1$

$$\begin{pmatrix} x_{-k-1} \\ p_{-k-1} \end{pmatrix} := \begin{pmatrix} x_{-k} \\ p_{-k} \end{pmatrix} - h\int_0^1 \begin{pmatrix} \frac{\partial H}{\partial p}(x(r), p(r))^{\mathrm{T}} \\ -\frac{\partial H}{\partial x}(x(r), p(r))^{\mathrm{T}} \end{pmatrix} \mathrm{d}r$$

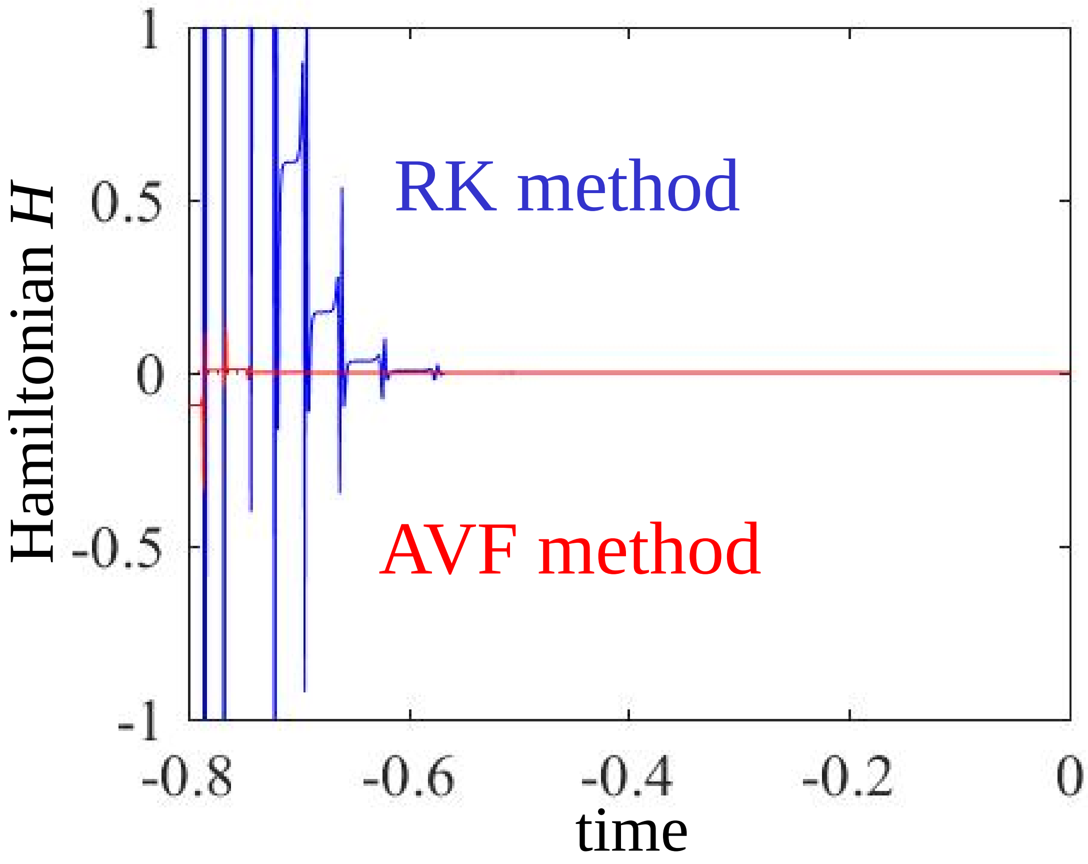
where $x(r) = (1-r)x_{-k} + rx_{-k-1},\ \ p(r) = (1-r)p_{-k} + rp_{-k-1}$

- Comput. of the integral $\cdots$ Gaussian quadrature

$$\int_0^1 \begin{pmatrix} \frac{\partial H}{\partial p}(x(r), p(r))^{\mathrm{T}} \\ -\frac{\partial H}{\partial x}(x(r), p(r))^{\mathrm{T}} \end{pmatrix} \mathrm{d}r \approx \sum_{i=1}^{\ell} w_i \begin{pmatrix} \frac{\partial H}{\partial p}(x(r_i), p(r_i))^{\mathrm{T}} \\ -\frac{\partial H}{\partial x}(x(r_i), p(r_i))^{\mathrm{T}} \end{pmatrix}$$

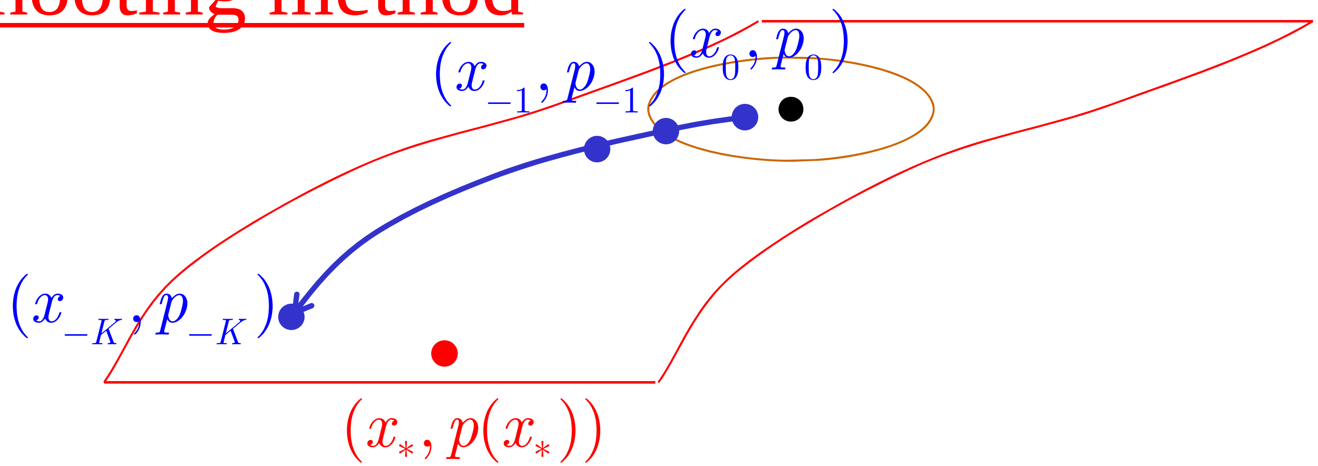
- Need to solve a nonlinear eq. for (x_{-k-1}, p_{-k-1})

Example



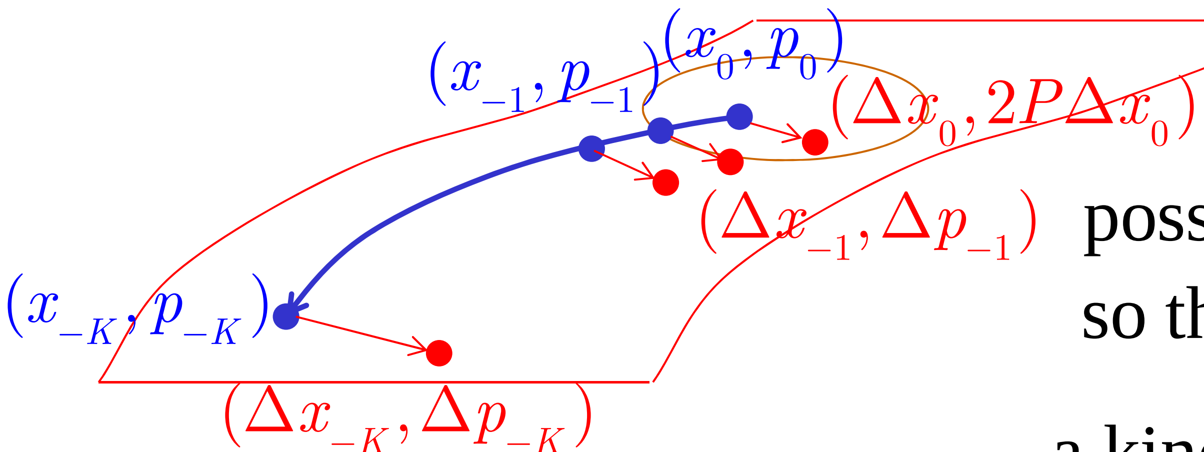
- AVF method keeps the Hamiltonian constant
- It does not necessarily mean its superiority

4. Shooting method



How to choose an initial point that leads to a given x_* ?

- Idea



Evolution of the perturb. is linear

possible to choose Δx_0 so that $\Delta x_{-K} = x_* - x_{-K}$

$\cdots$ a kind of Newton method

Time evolution of the perturbation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\Delta x\\ \Delta p\end{pmatrix} = \begin{pmatrix} \frac{\partial^2 H}{\partial p \partial x}(x(t),p(t)) & \frac{\partial^2 H}{\partial p^2}(x(t),p(t)) \\ -\frac{\partial^2 H}{\partial x^2}(x(t),p(t)) & -\frac{\partial^2 H}{\partial x \partial p}(x(t),p(t)) \end{pmatrix}\begin{pmatrix}\Delta x\\ \Delta p\end{pmatrix}$$

$$\begin{pmatrix}\Delta x_0\\ 2P\Delta x_0\end{pmatrix} = \begin{pmatrix} I \\ 2P\end{pmatrix}\Delta x_0 \xrightarrow{\text{numerically}} \begin{pmatrix}\Delta x_{-1}\\ \Delta p_{-1}\end{pmatrix} = \begin{pmatrix} U_{-1} \\ V_{-1}\end{pmatrix}\Delta x_0$$

$$\begin{pmatrix}\Delta x_{-K}\\ \Delta p_{-K}\end{pmatrix} = \begin{pmatrix} U_{-K} \\ V_{-K}\end{pmatrix}\Delta x_0$$

For $\Delta x_{-K} = U_{-K}\Delta x_0 = x_* - x_{-K}$

choose Δx_0

For a new initial pt. $(x_0 + \Delta x_0, p_0 + 2P\Delta x_0)$,

compute a sequence

Repeat until convergence

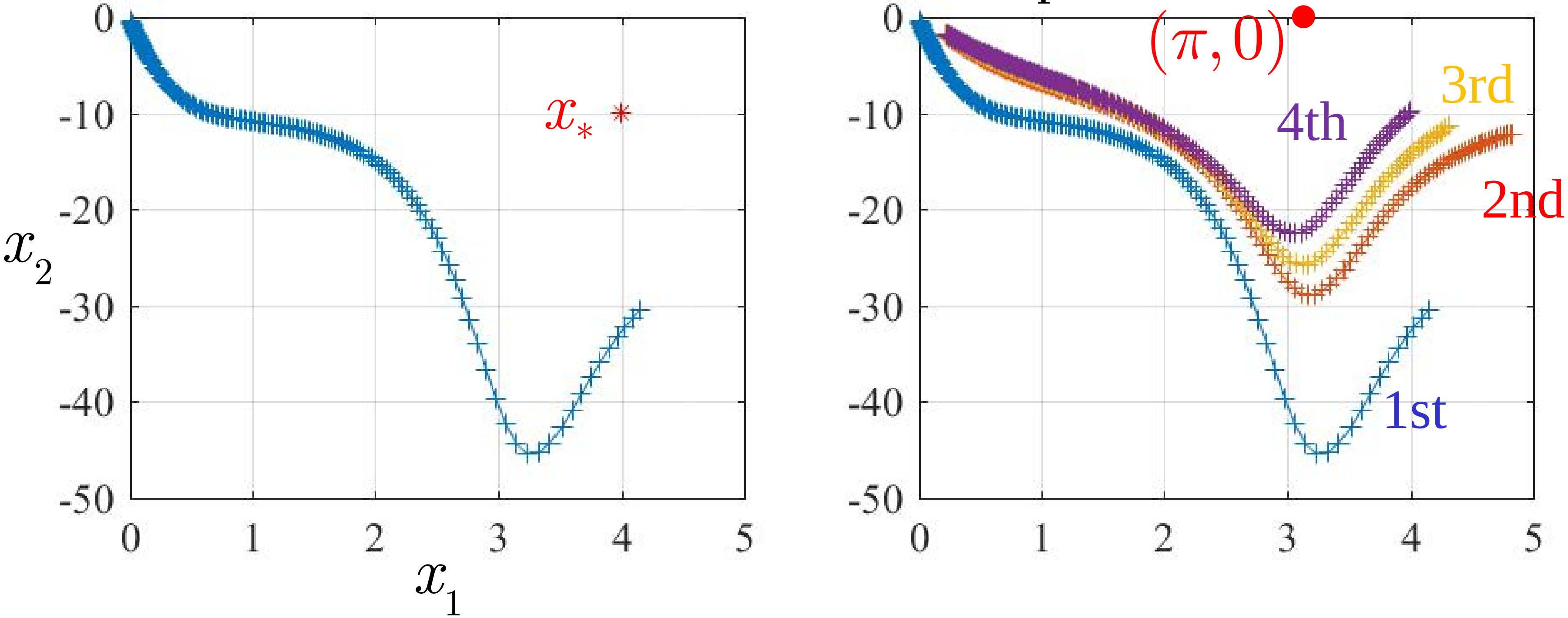
Example: swing-up of a pendulum

$$x_0 = (-0.01 \;\; 0.05)^{\mathrm{T}}$$

$$h = 0.002, \quad -Kh = -0.45$$

$$x_* = (4 \;\; -10)^{\mathrm{T}}$$

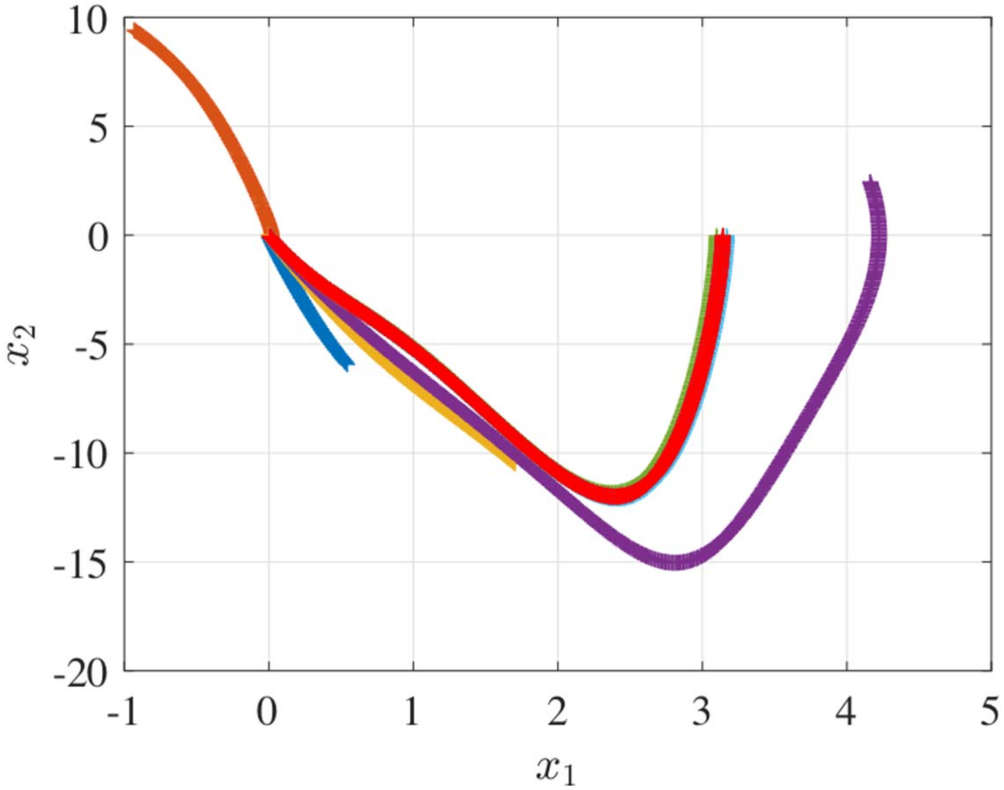
computational time: 52 s



- Trajectory for the swing up can be computed

Swing-up with multiple swings

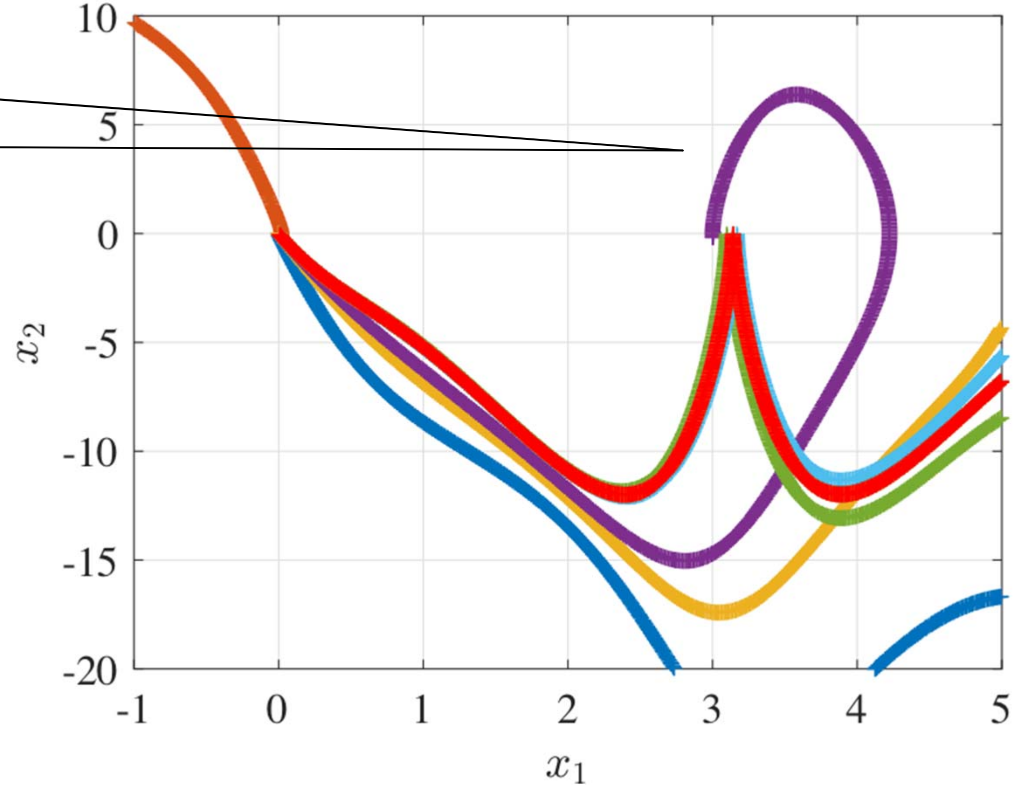
1. Compute trajectories for 1 swing



Swing-up with multiple swings

1. Compute trajectories for 1 swing
2. Extend the simulation time for each trajectory

Find a trajectory close
to 2 swing

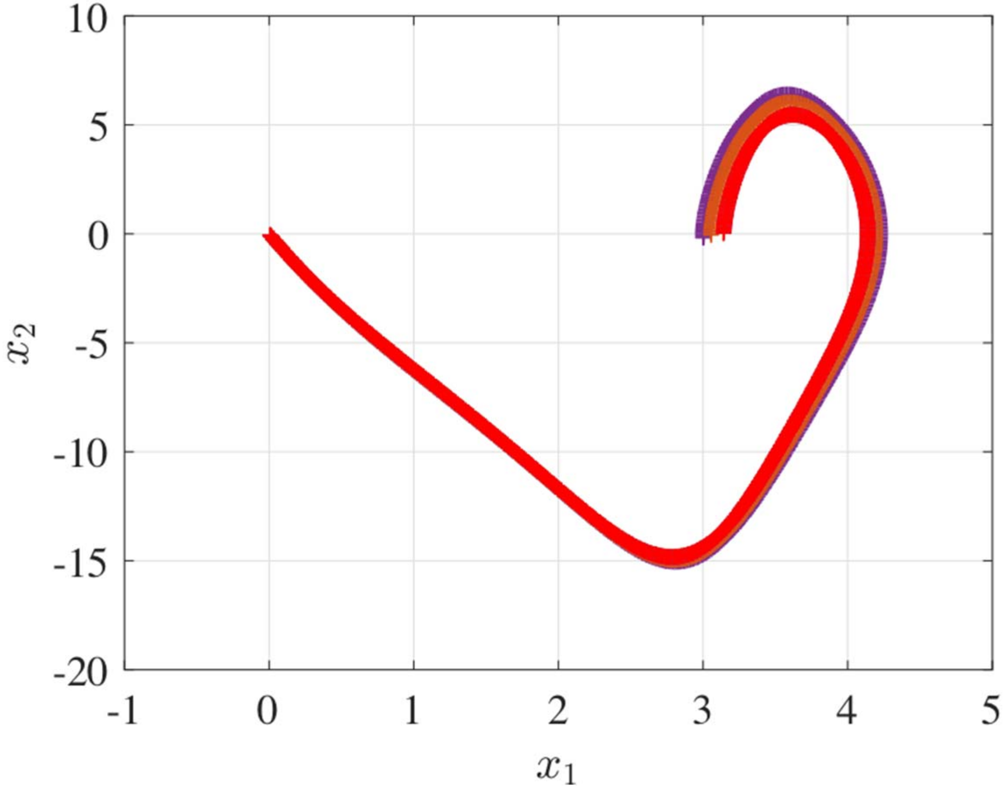


Swing-up with multiple swings

1. Compute trajectories for 1 swing

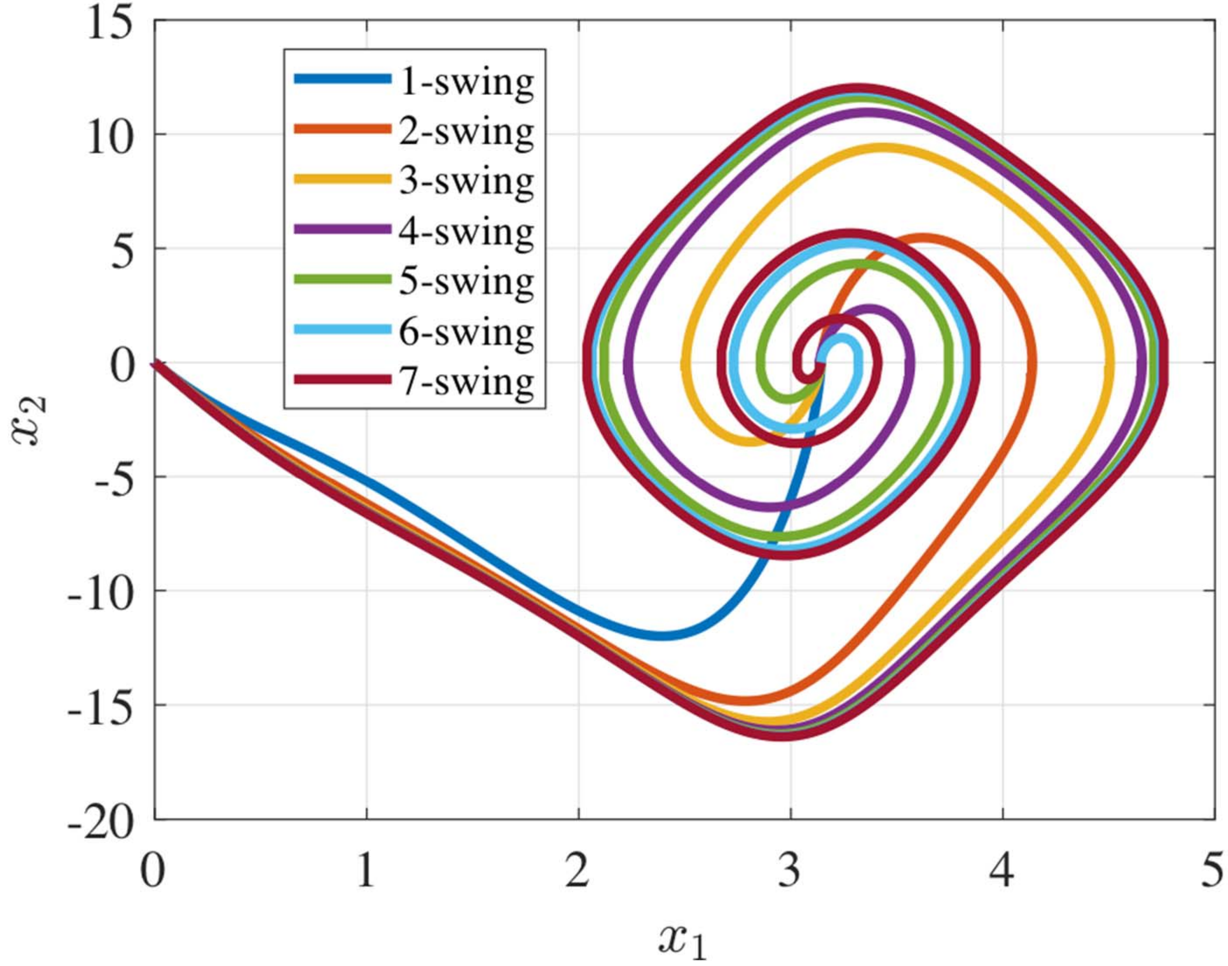
2. Extend the simulation time for each trajectory

3. Run the shooting method from the trajectory



Swing-up with 1 to 7 swings

cf. [Horibe--Sakamoto 17]

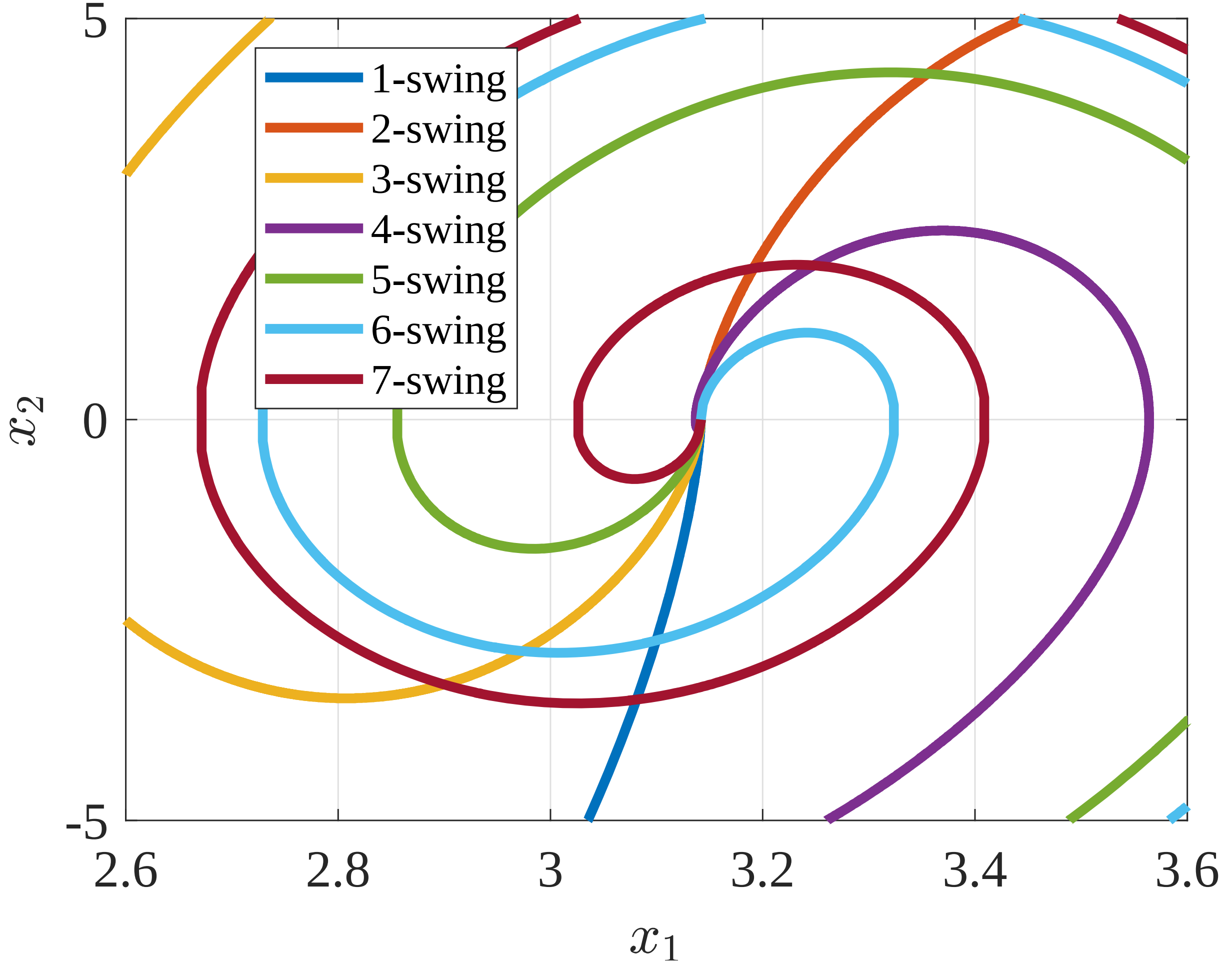


Swing-up with 1 to 7 swings

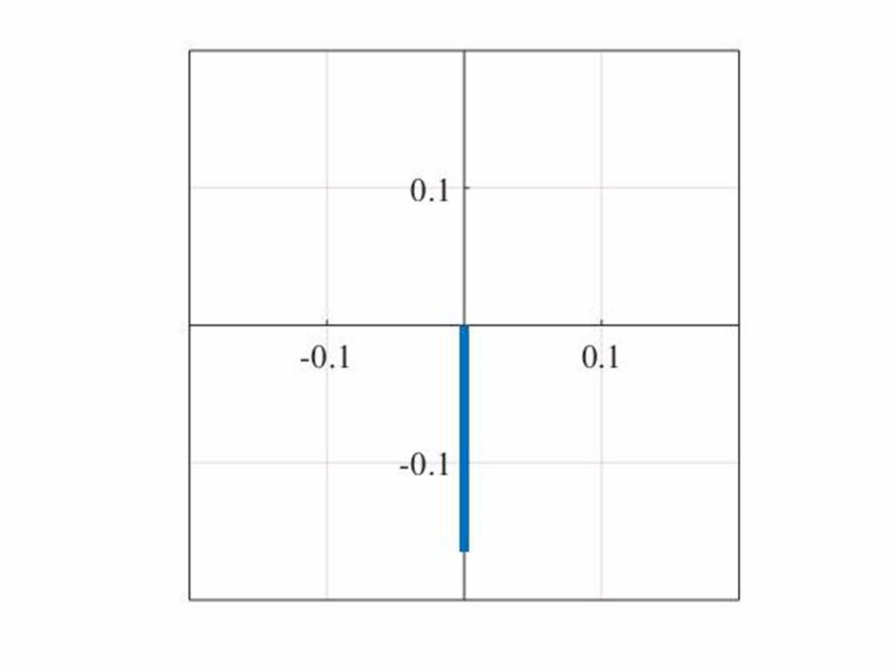
cf. [Horibe--Sakamoto 17]

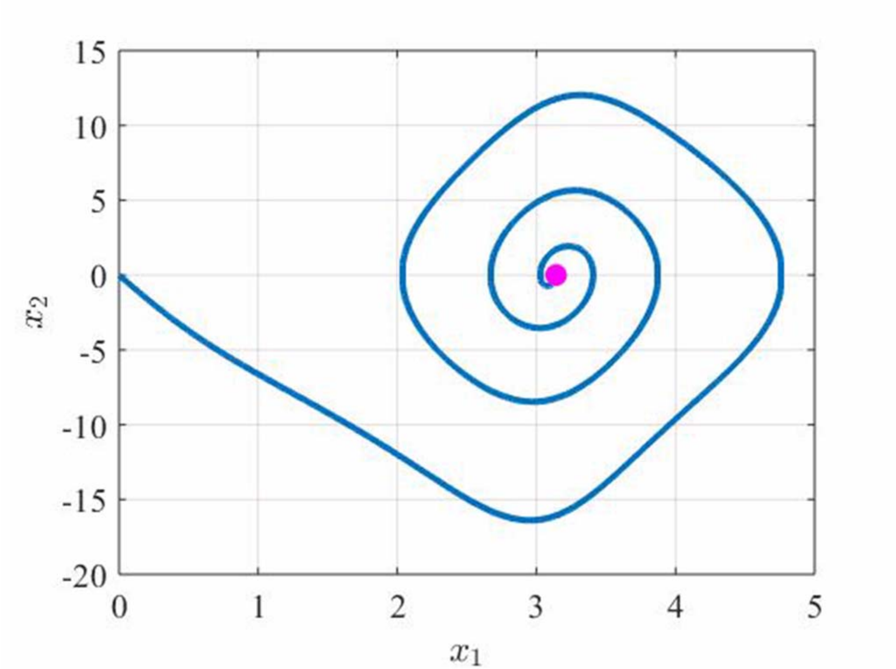
Obj. func.

$J_1 = 0.0073335$
$J_2 = 0.0033514$
$J_3 = 0.0025836$
$J_4 = 0.0023331$
$J_5 = 0.0022350$
$J_6 = 0.0022133$
$J_7 = 0.0022183$

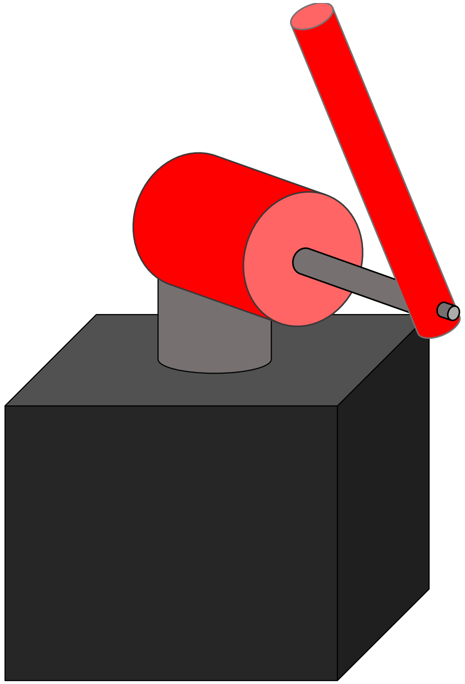


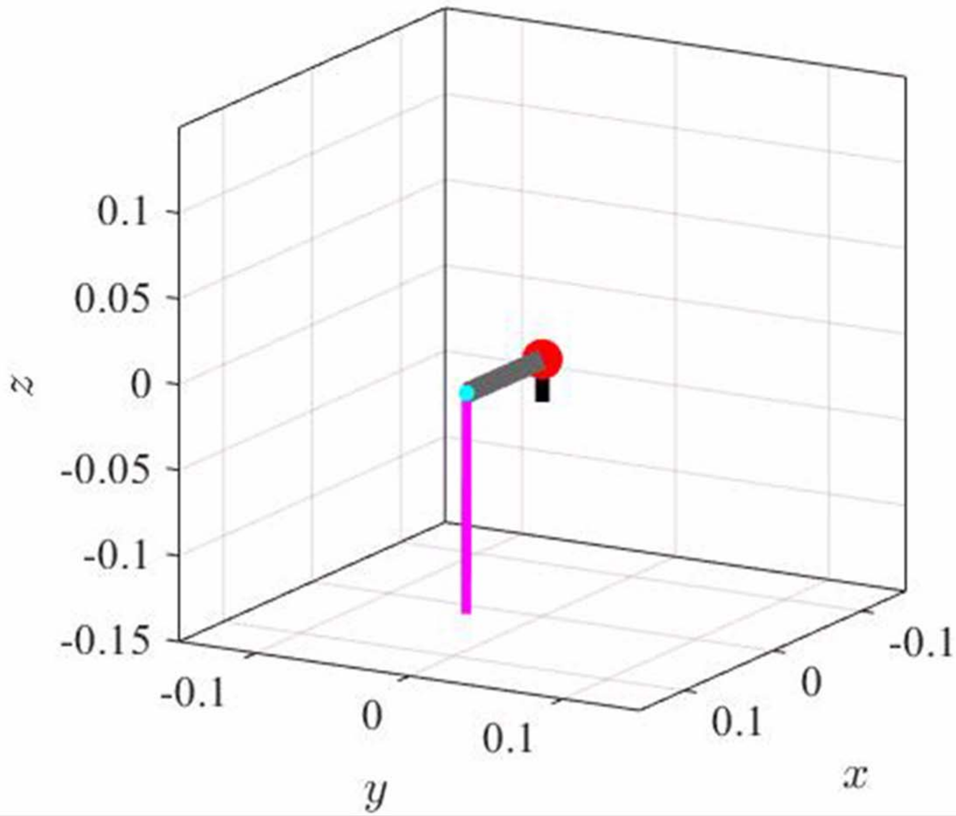
Swing-up with 7 swings





Application to the Furuta pendulum





- Trajectory with 3 swings is successfully computed

5. Summary

Application of numerical computational techniques to the stable-manifold method

- Structure-preserving numerical methods for stable computation of a Hamiltonian system
- Shooting method for systematic choice of an initial pt.
- Reduction of know-how factors of the stable-manifold method