Geometry of LMIs and determinantal representations of algebraic plane curves

Didier HENRION
LAAS-CNRS Toulouse, FR
Czech Tech Univ Prague, CZ

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Outline

1. LMI optimization

2. Geometry of LMI sets

3. Rational curves

4. Cubics

5. Dixon's construction

LMI

Linear matrix inequality

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0$$

where F_i are given symmetric real matrices and constraint $\succeq 0$ means positive semidefinite (all eigenvalues real nonnegative)

Arise in control theory (Lyapunov 1890, Willems 1971, Boyd et al. 1994), combinatorial optimization, finance, structural mechanics, and many other areas

Key property = convex in x

Semidefinite programming

Decision problem

$$\begin{array}{ll} \text{min}_x & \sum_i c_i x_i \\ \text{s.t.} & F_0 + \sum_i x_i F_i \succeq 0 \end{array}$$

Optimization over LMIs = semidefinite programming, versatile generalization of linear (and convex quadratic) programming to the convex cone of positive semidefinite matrices

At given accuracy can be solved in polynomial time using interior-point methods (Nesterov, Nemirovski 1994)

Many public-domain solvers available

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Geometry of LMI sets

How does an LMI set

$$\mathcal{F} = \{x \in \mathbb{R}^n \,:\, F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0\}$$

look like in Euclidean space ?

Matrix $F(x)$ is PSD iff its diagonal minors $f_i(x)$ are nonnegative

So the LMI set can be described as

$$\mathcal{F} = \{x \in \mathbb{R}^n \,:\, f_i(x) \geq 0,\ i = 1, 2, \ldots\}$$

a convex closed basic semialgebraic set

Semialgebraic formulation

For an d-by-d matrix $F(x)$ we have $2^d - 1$ diagonal minors

A simpler criterion follows from the fact that a poly $t \mapsto f(t) = \sum_k f_{d-k} t^k = \prod_k (t - t_k)$ which has only real roots satisfies $t_k \leq 0$ iff $f_k \geq 0$

Apply to characteristic poly $f(t,x) = \det(tI_d + F(x)) = \sum_{k=0}^{d} f_{d-k}(x) t^k$ which is monic, i.e. $f_0(x) = 1$

Only d poly ineqs $f_k(x) \geq 0$ to be checked

Polys $f_k(x)$ are sums of principal minors of $F(x)$ of order k or equivalently sums of k-term-products of eigenvalues of $F(x)$

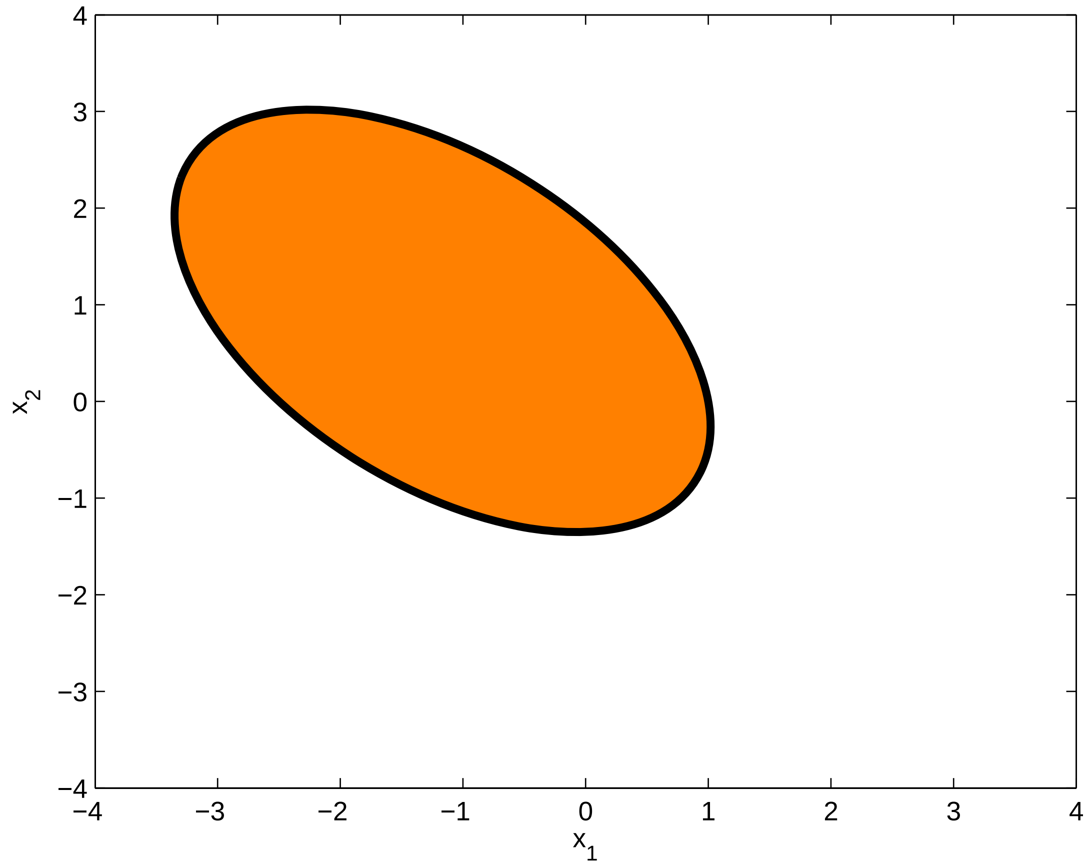
Example of 2D LMI feasible set

$$F(x) = \begin{bmatrix} 1 - x_1 & x_1 + x_2 & x_1 \\ x_1 + x_2 & 2 - x_2 & 0 \\ x_1 & 0 & 1 + x_2 \end{bmatrix} \succeq 0$$

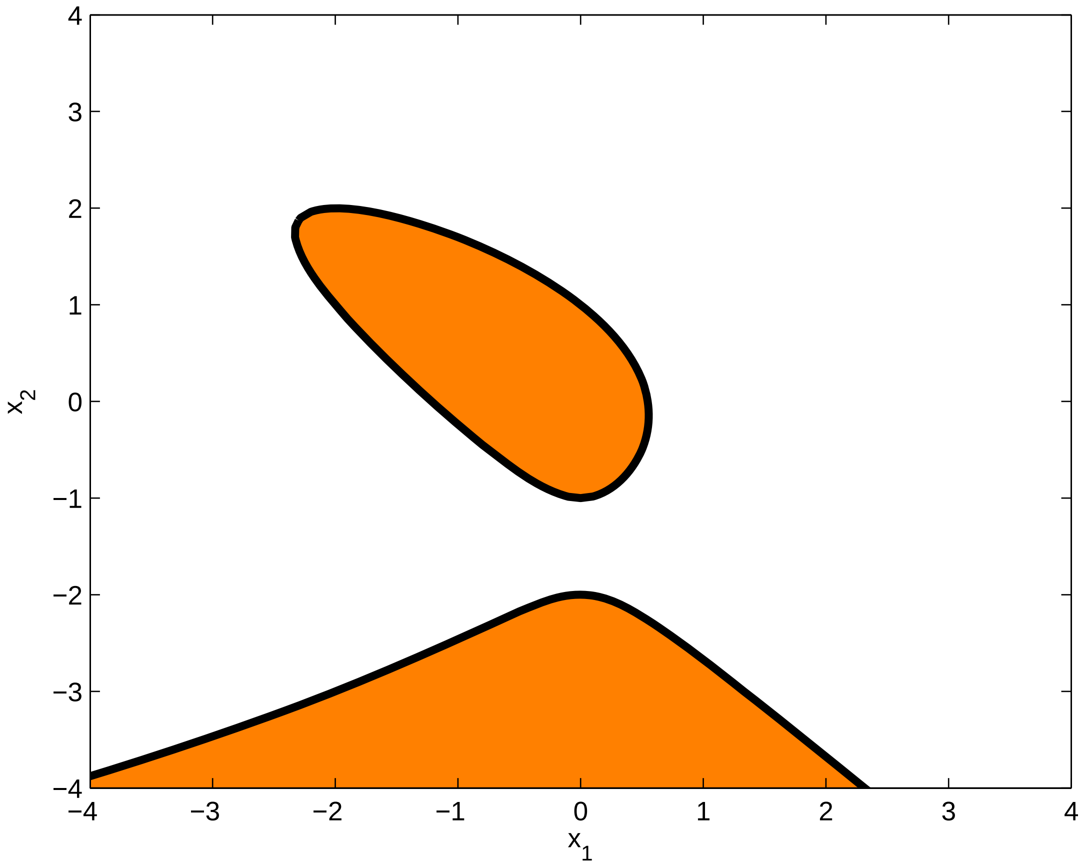
System of 3 polynomial inequalities $f_i(x) \geq 0$

$$f_1(x) = \text{trace}\, F(x) = 4 - x_1 \geq 0$$

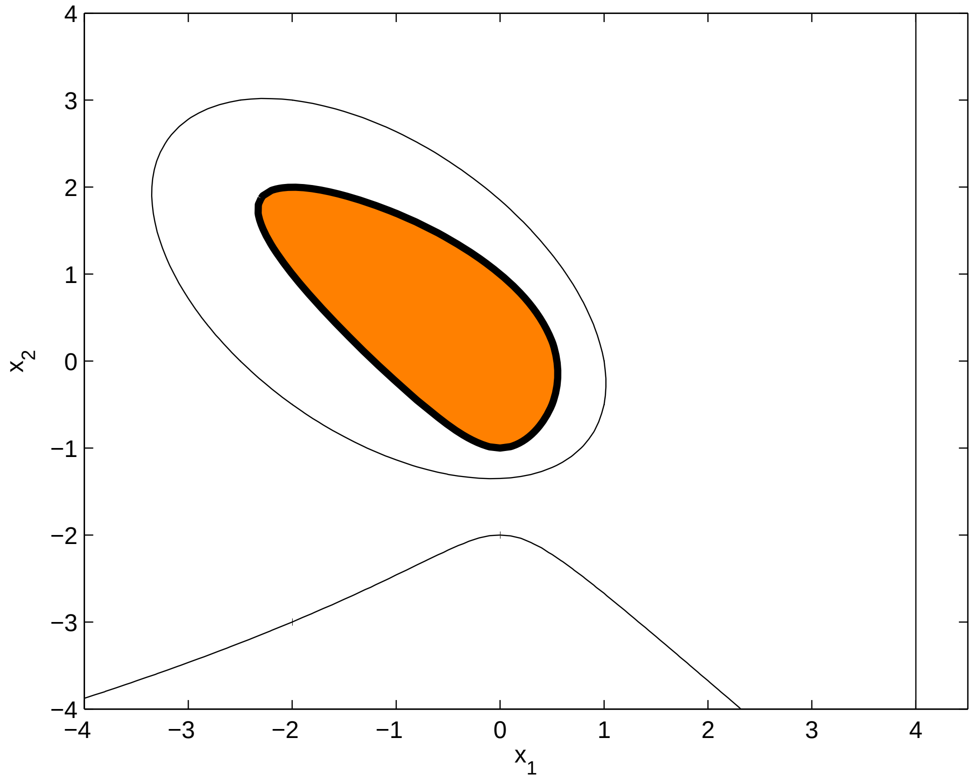
$$f_2(x) = 5 - 3x_1 + x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2 \geq 0$$



$$f_3(x) = \det F(x) = 2 - 2x_1 + x_2 - 3x_1^2 - 3x_1x_2 - 2x_2^2 - x_1x_2^2 - x_2^3 \geq 0$$

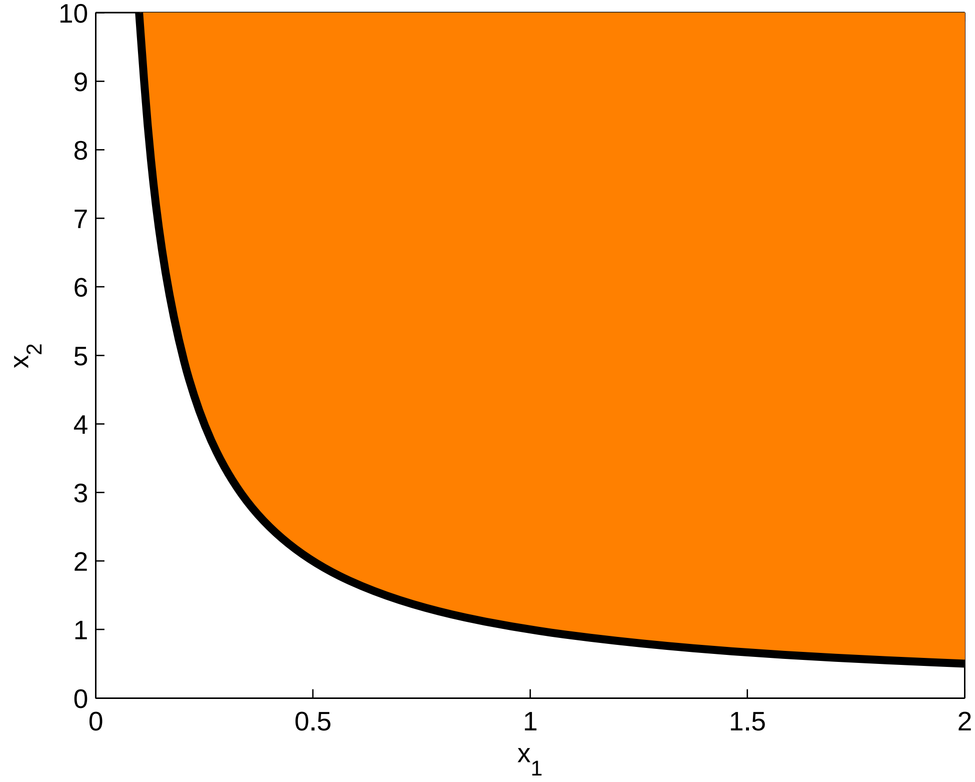


LMI set = intersection of level-sets $f_k(x) \geq 0$, $k = 1, 2, 3$



Boundary of LMI region shaped by determinant
Other polys only isolate convex connected component

LMI set or not ?



$$x_1 x_2 \geq 1 \text{ and } x_1 \geq 0$$

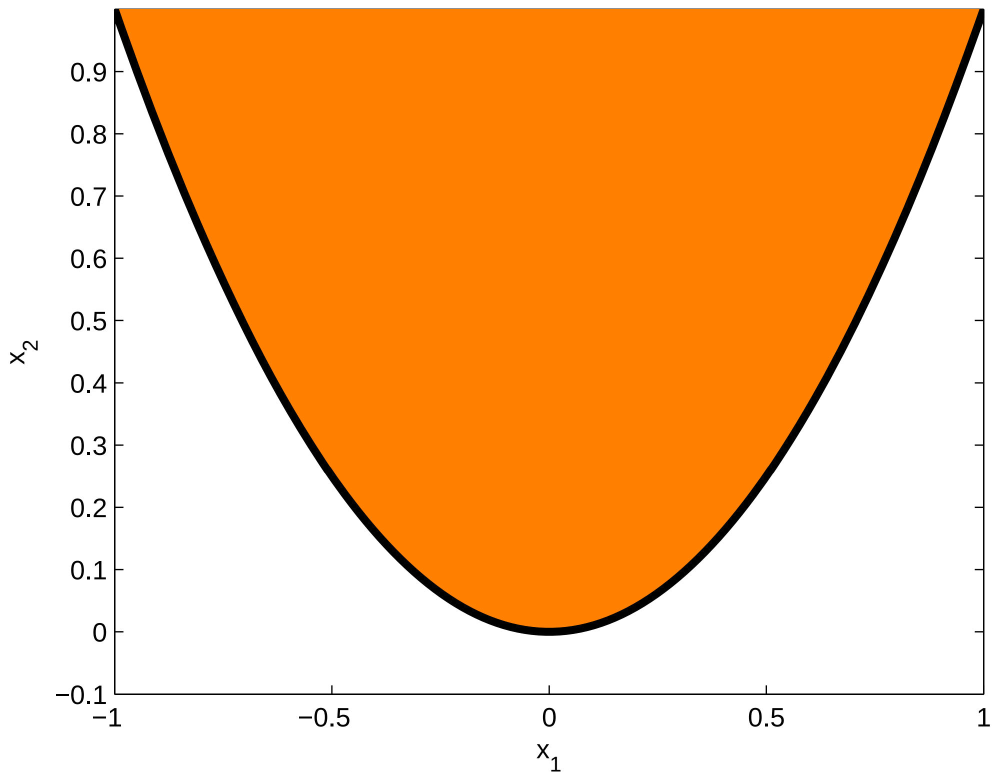
LMI

$$x_1 x_2 \geq 1 \text{ and } x_1 \geq 0$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} x_1 & 1 \\ 1 & x_2 \end{bmatrix} \succeq 0$$

LMI set or not ?



$$x_2 \geq x_1^2$$

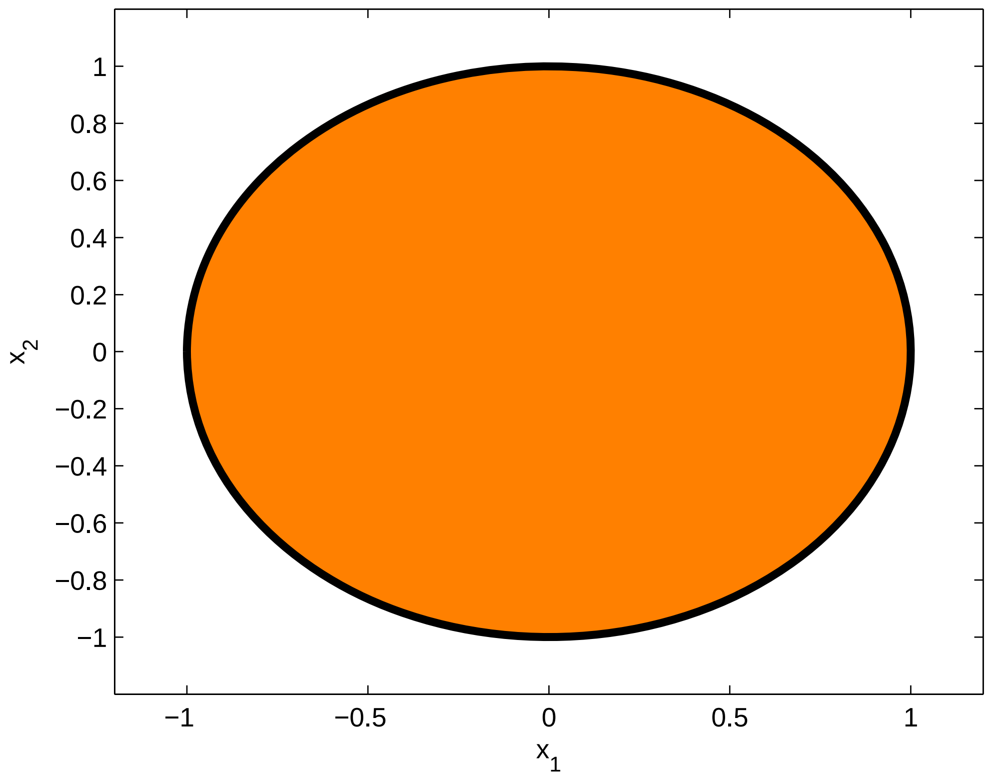
LMI

$$x_2 \geq x_1^2$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} 1 & x_1 \\ x_1 & x_2 \end{bmatrix} \succeq 0$$

LMI set or not ?



$$x_1^2 + x_2^2 \leq 1$$

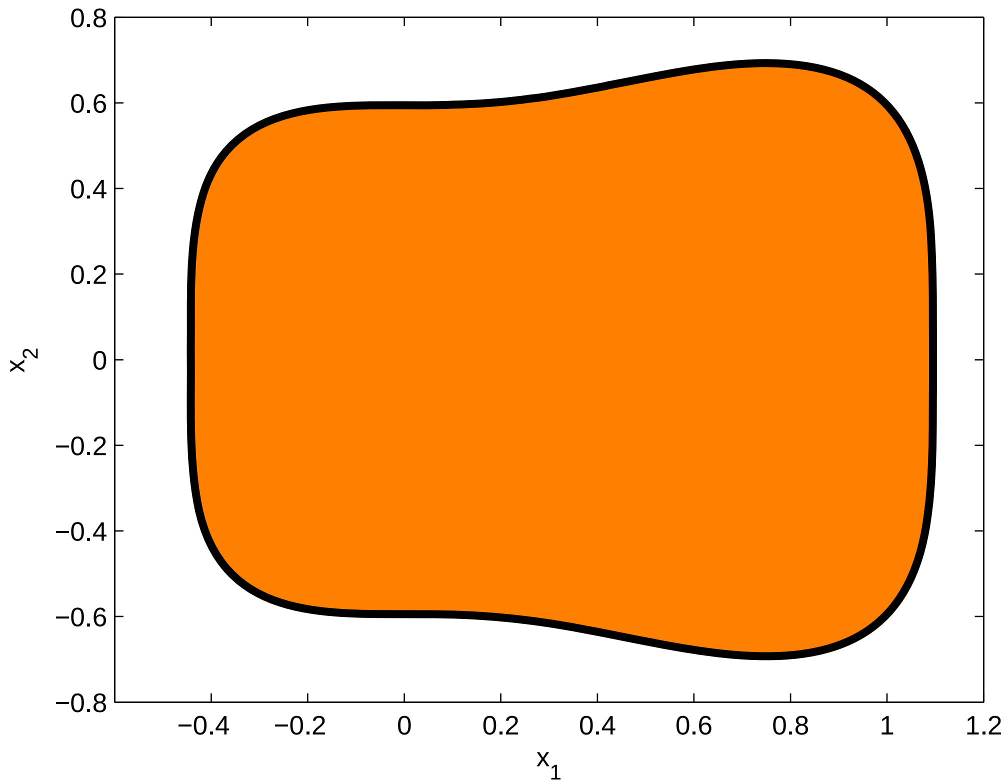
LMI

$$x_1^2 + x_2^2 \leq 1$$

$$\Longleftrightarrow$$

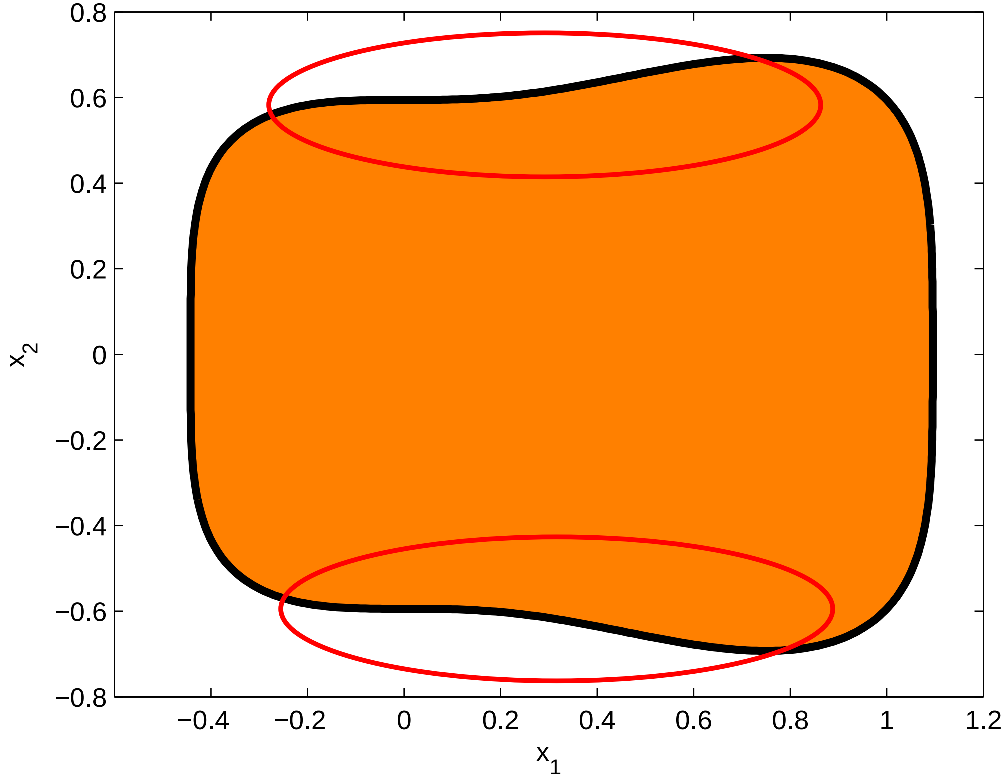
$$\begin{bmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{bmatrix} \succeq 0$$

LMI set or not ?



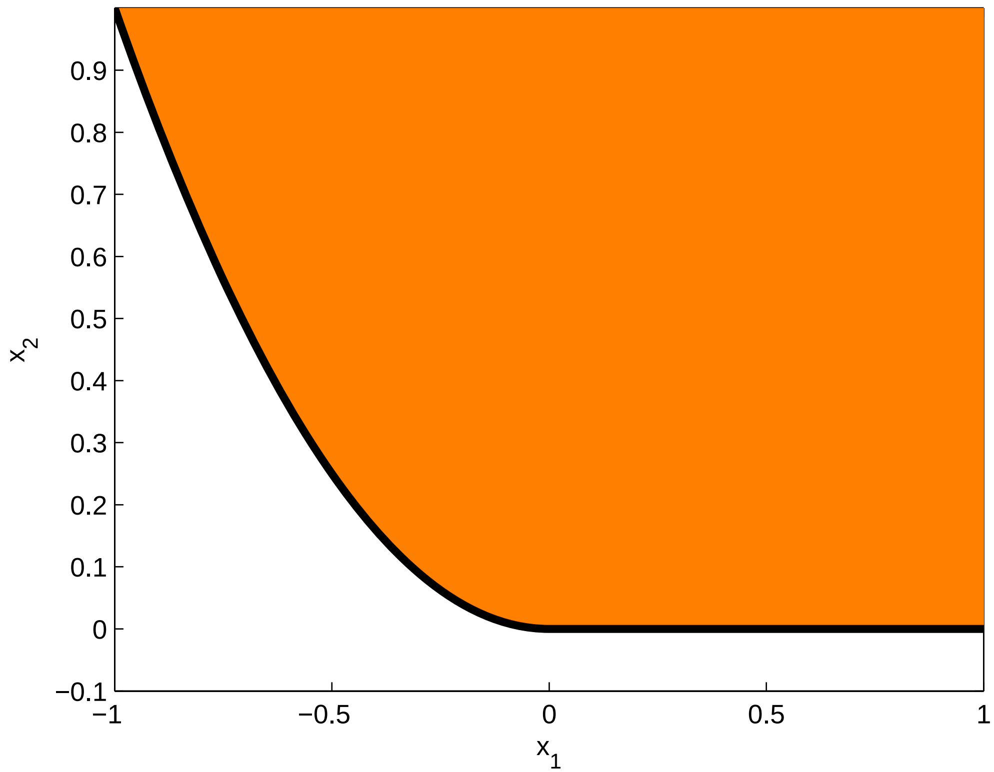
$$x_1^4 + x_2^4 - x_1^3 - \frac{1}{8} \leq 0$$

NOT LMI: not convex



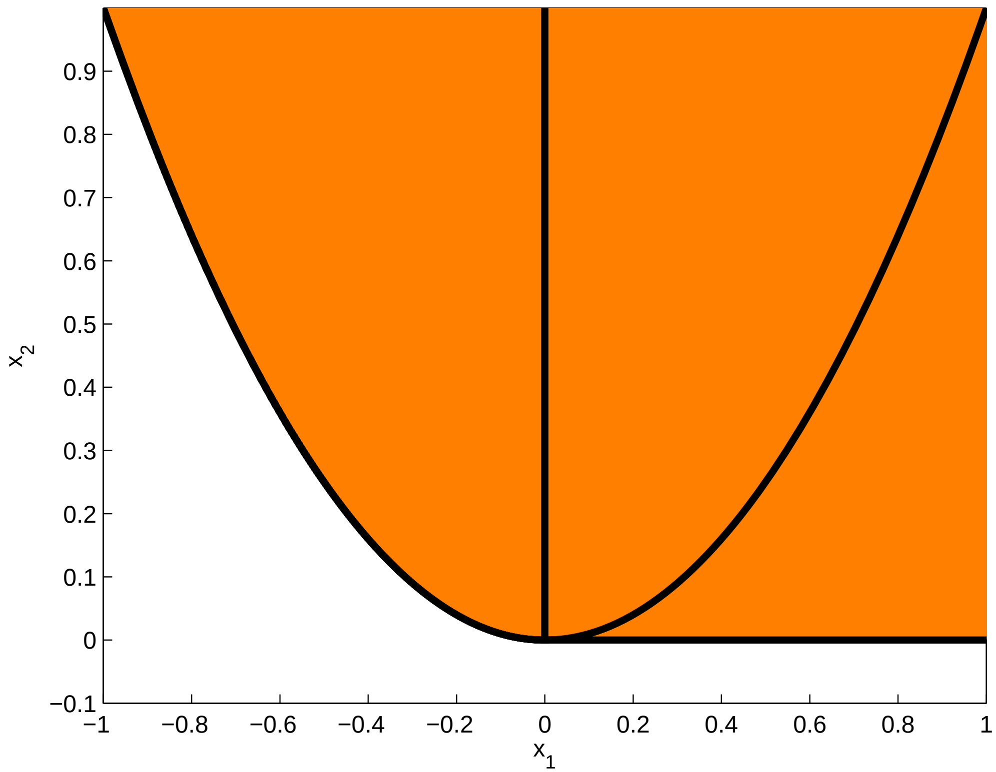
$$x_1^4 + x_2^4 - x_1^3 - \frac{1}{8} \leq 0$$

LMI set or not ?



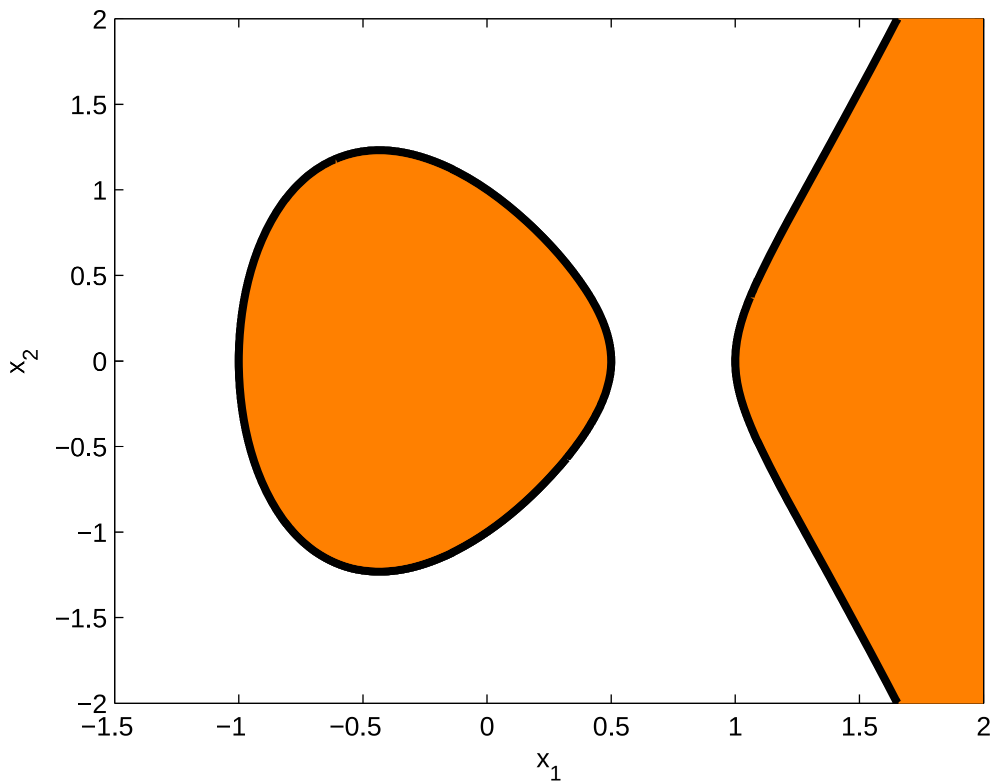
$$\{x \in \mathbb{R}^2 \;:\; t^2 + 2x_1 t + x_2 \geq 0,\ \forall t \in \mathbb{R}\}$$

NOT LMI: not basic semialgebraic



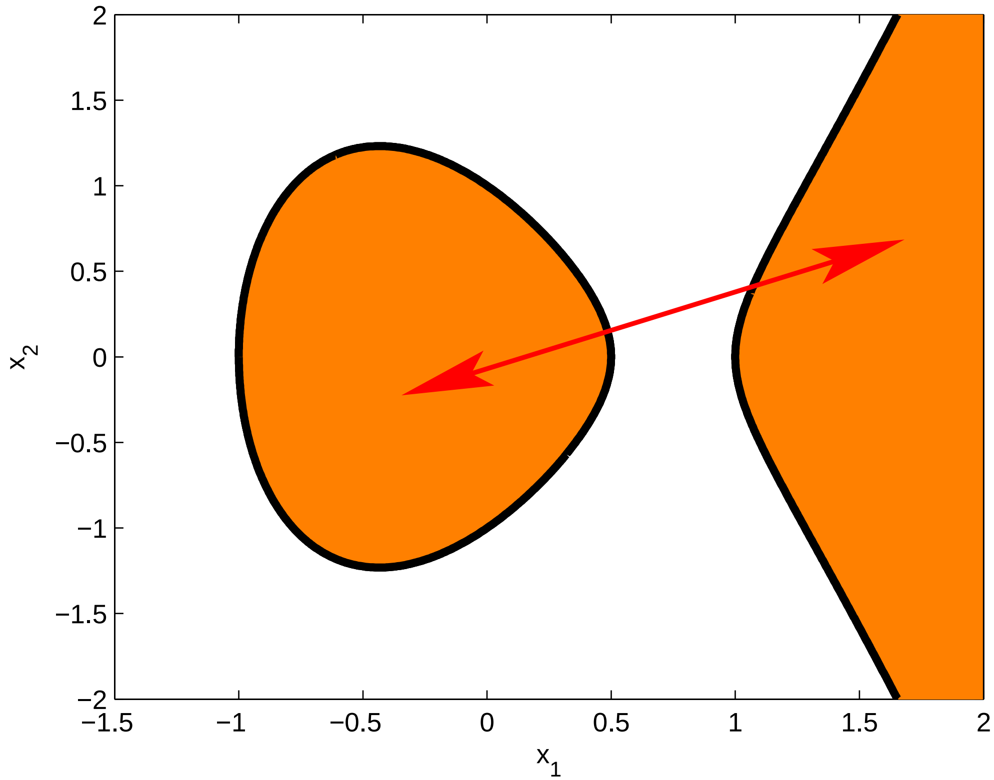
$$x_2 \geq x_1^2 \text{ or } x_1, x_2 \geq 0$$

LMI set or not ?



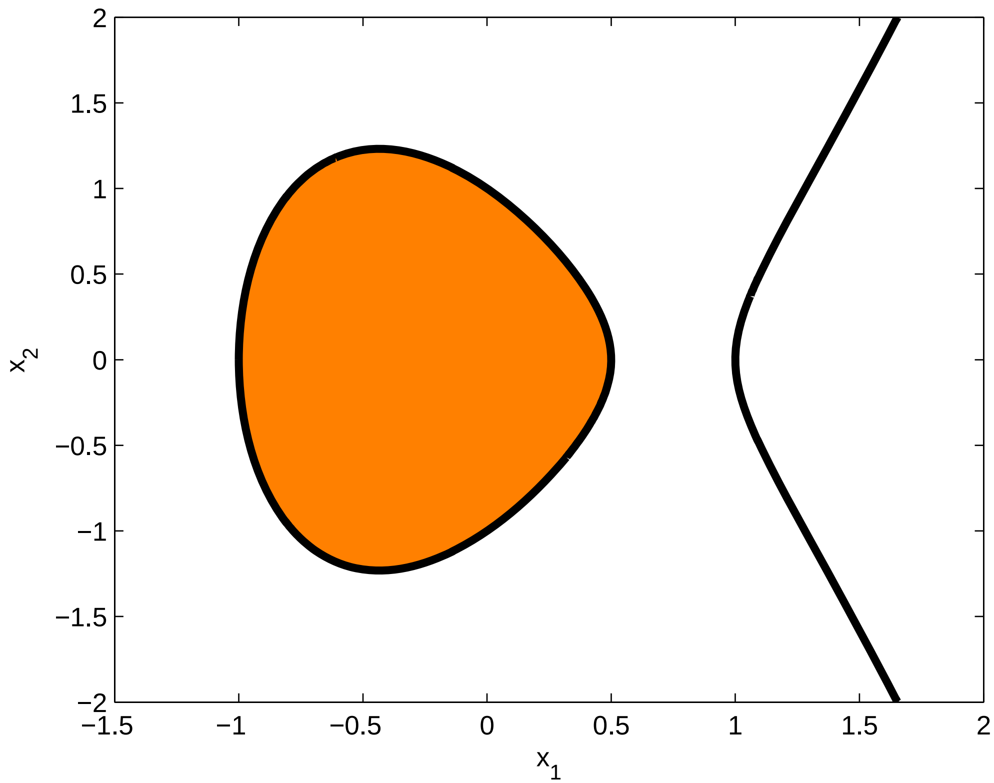
$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0$$

NOT LMI: not connected



$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0$$

LMI set or not ?



$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0 \text{ and } x_1 \leq \frac{1}{2}$$

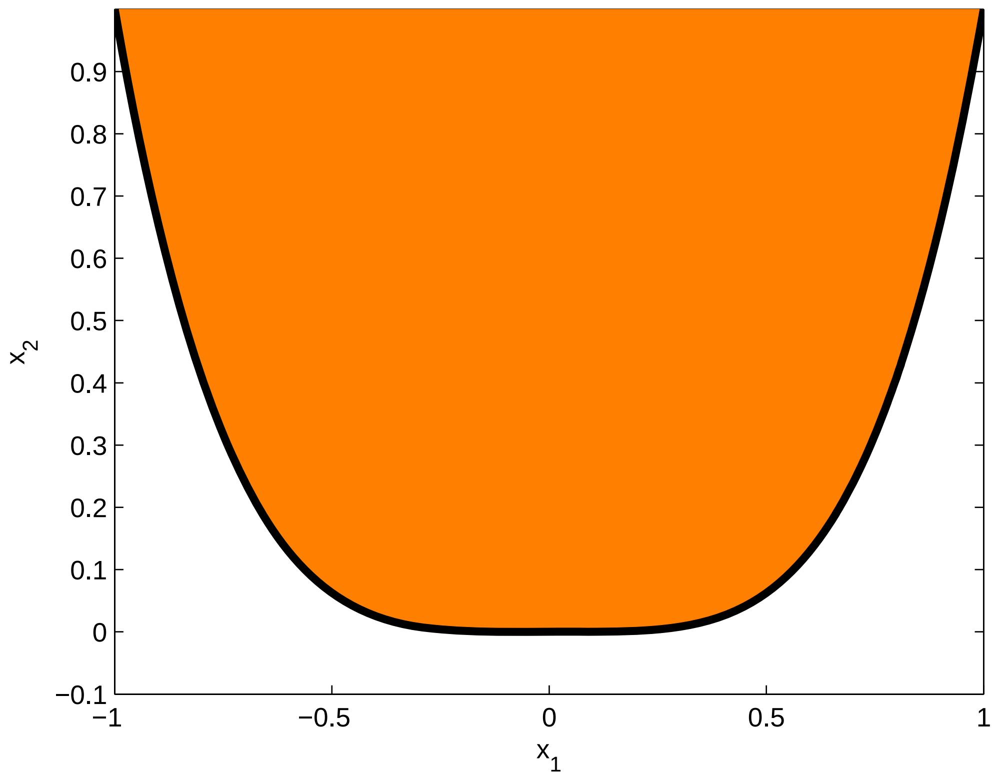
LMI

$$1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 \geq 0 \text{ and } x_1 \leq \tfrac{1}{2}$$

$$\Longleftrightarrow$$

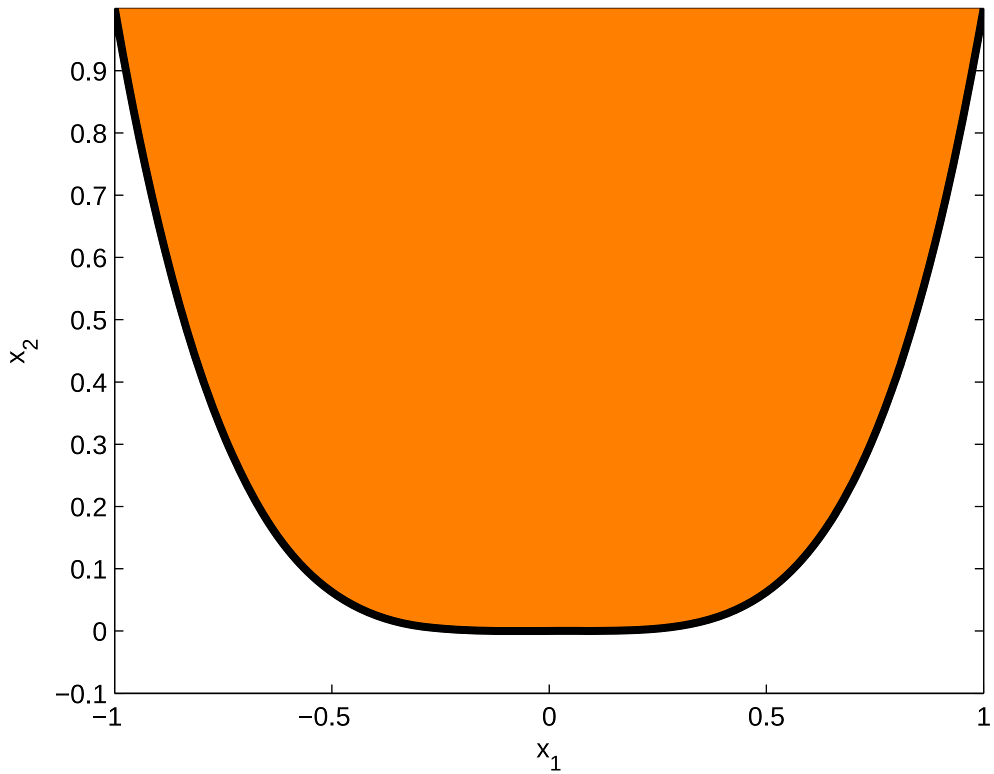
$$\begin{bmatrix} 1 & x_1 & 0 \\ x_1 & 1 & x_2 \\ 0 & x_2 & 1 - 2x_1 \end{bmatrix} \succeq 0$$

LMI set or not ?



$$x_1^4 + x_2^4 \leq 1$$

NOT LMI: not rigidly convex



$$x_1^4 + x_2^4 \leq 1$$

Determinantal representation

Consider the non-empty semialgebraic set

$$\mathcal{F} = \{x \in \mathbb{R}^n \, : \, f(x) \geq 0\}$$

where $f(x)$ is a given polynomial of degree d

Without loss of generality, assume that we are given a point e (typically the origin) satisfying $f(e) = 1$

Since the boundary of an LMI set is shaped by a determinant, can we find symmetric real matrices F_i such that

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i, \quad \det F(x) = f(x)$$

So we would like to find a linear symmetric determinantal representation for polynomial $f(x)$

Definite determinantal representation = LMI

Once we have $\det F(x) = f(x)$, we would like to know whether

$$\begin{array}{rcl} \mathcal{F} & = & \text{closure}\,\{x \in \mathbb{R}^n \;:\; \det F(x) > 0\} \ni e \\ & = & \{x \in \mathbb{R}^n \;:\; F(x) \succeq 0\} \end{array}$$

Since $f(e) = 1$, it holds $e \in \text{int}\,\mathcal{F}$ and $F(e) \succ 0$ so the representation must be definite for $\mathcal{F}$ to be expressed as an LMI

Under which conditions on f can we find such a definite representation ?

Define the algebraic curve

$$\mathcal{C} = \{x \in \mathbb{R}^n \;:\; f(x) = 0\}$$

containing the boundary of $\mathcal{F}$

Rigid convexity

Necessary condition for $\mathcal{F}$ to have a definite symmetric linear determinantal, or LMI representation:

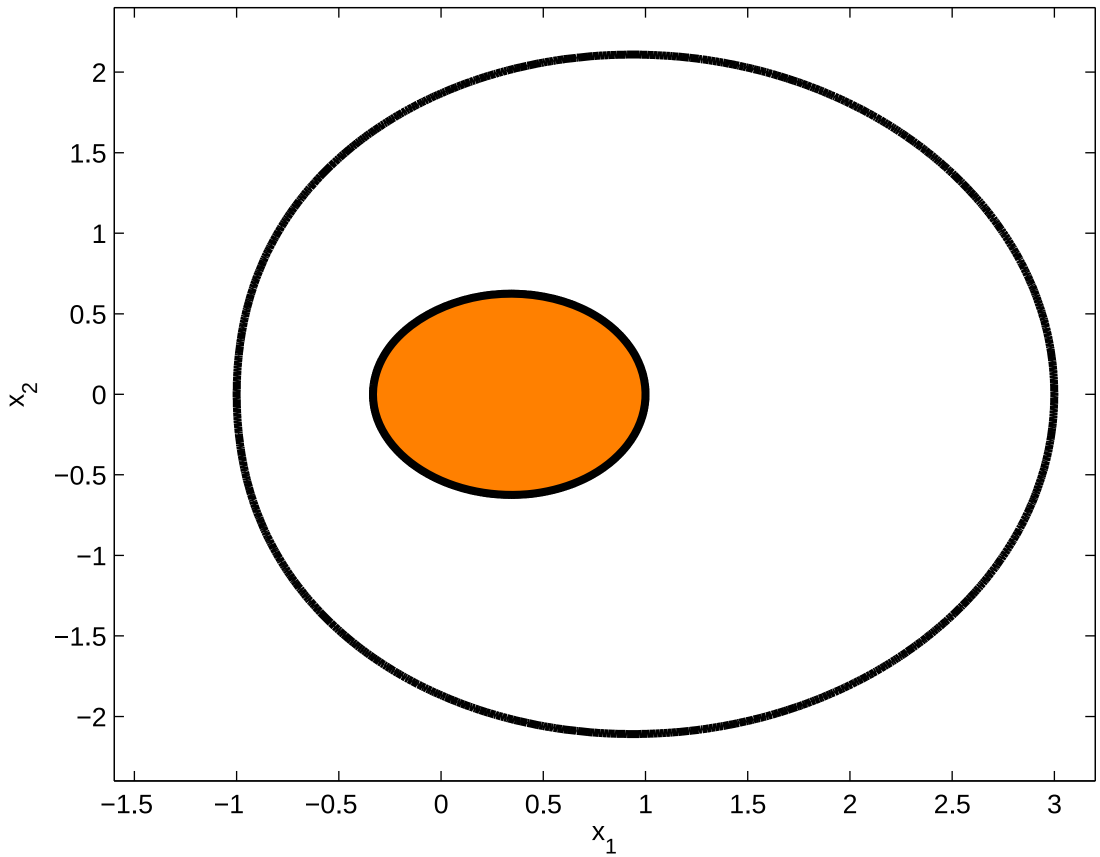
Any line passing through an interior point of $\mathcal{F}$ must intersect $\mathcal{C}$ exactly d times (counting multiplicities and points at infinity)

Rigid convexity implies convexity

Strong result by Helton and Vinnikov (2002): the condition is also sufficient in the plane, i.e. for $n = 2$

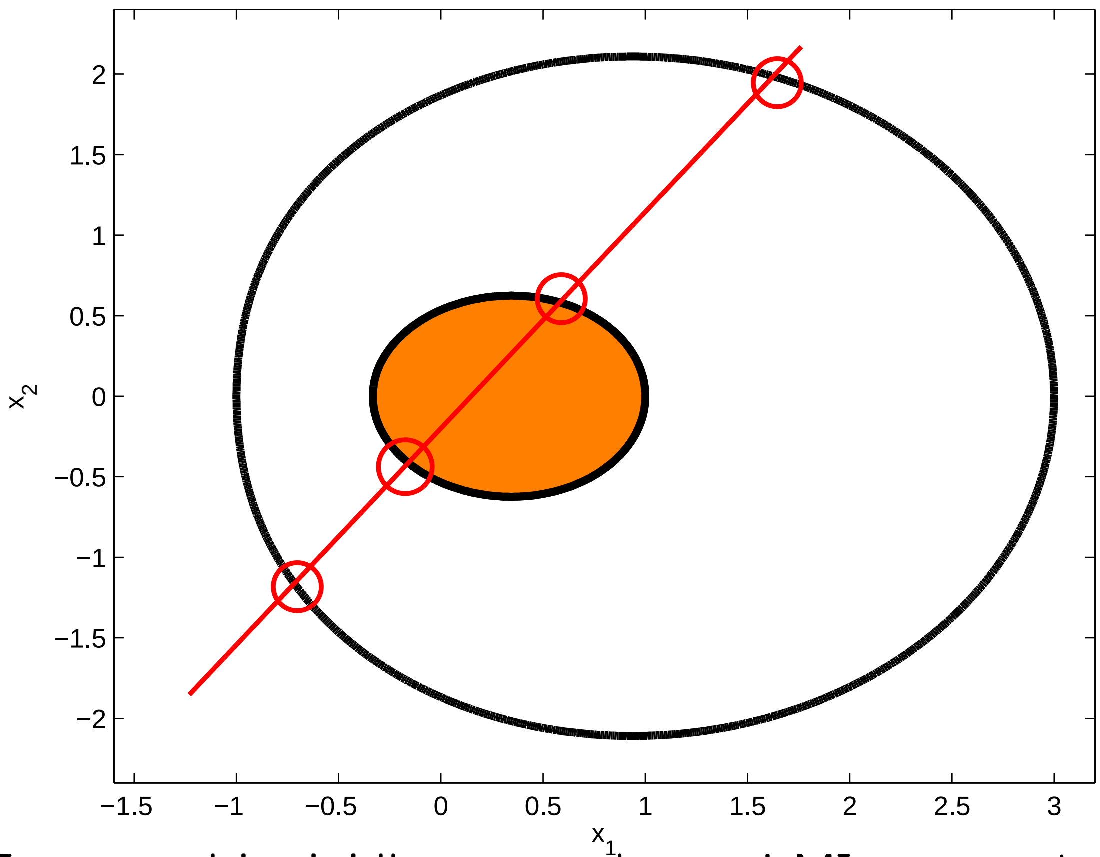
Also sufficient for $n > 2$?

Cartesian ovals



$$(3((x_1+1)^2+x_2^2+1)-10(x_1+1))^2-10((x_1+1)^2+x_2^2+1)+12(x_1+1)+1 \geq 0$$

Cartesian ovals



Inner oval is rigidly convex hence LMI representable

Constructive methods

Checking rigid convexity amounts to checking positive semidefiniteness of the Hermite matrix of polynomial $p(x)$ for all x

Given $f(x)$ and e, once we know that the set

$$\mathcal{F} = \{x \in \mathbb{R}^n \; : \; f(x) \geq 0\} \ni e$$

is rigidly convex, how can we systematically build symmetric matrices F_i such that

$$\mathcal{F} = \{x \in \mathbb{R}^n \; : \; F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0\}$$

and so $f(x) = \det F(x)$? When/how can we enforce $F_0 = I$?

In the sequel we focus exclusively on the plane case $(n = 2)$

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Rational curves

An algebraic plane curve of genus zero, that is, with a maximal number of singularities

$$\{x \in \mathbb{R}^2 \; : \; f(x) = 0\}$$

admits a rational parametrization

$$x_1(t) = \frac{f_1(t)}{f_0(t)}, \quad x_2(t) = \frac{f_2(t)}{f_0(t)}$$

with $f_i(t)$ real polys of real indeterminate t

Degrees of f_i do not exceed degree of f

Coeffs of f_i chosen in (typically small) algebraic extension of the coeff field of f

Bezoutian

Determinantal representation follows from the resultant of the two polys

$$\begin{aligned} g_1(t, x_1) &= f_0(t) - x_1(t) f_1(t) \\ g_2(t, x_2) &= f_0(t) - x_2(t) f_2(t) \end{aligned}$$

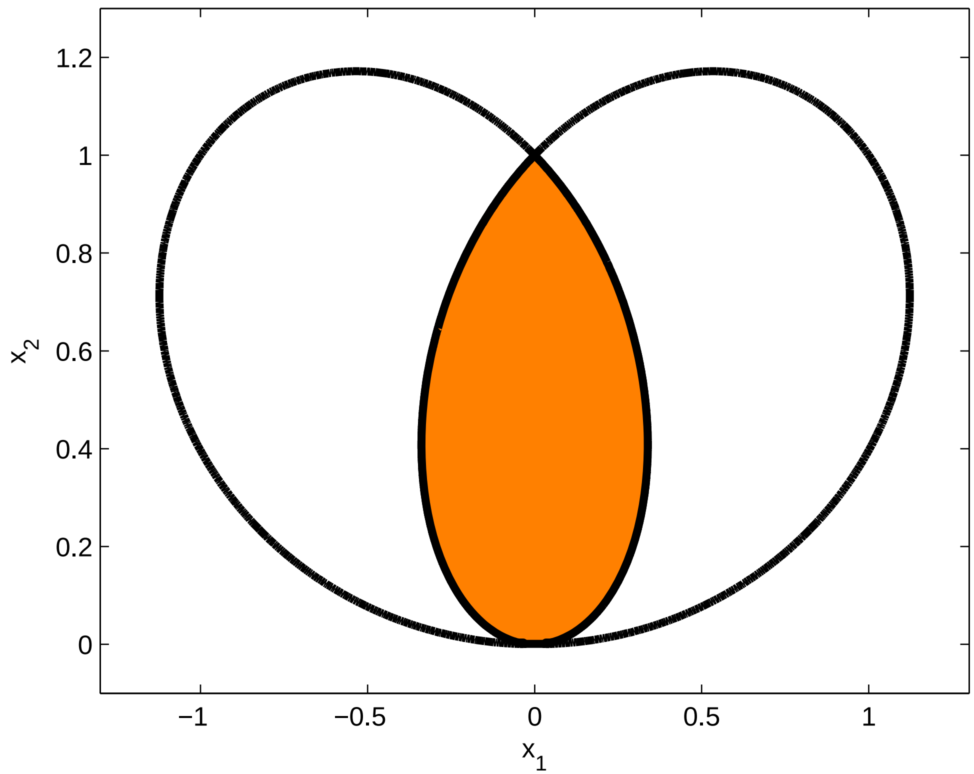
with respect to t (variable to be eliminated)

Bezout matrix $B_t(g_1, g_2)$ is symmetric and linear in x such that

$$\det B_t(g_1, g_2) = f(x)$$

hence $F(x) = B_t(g_1, g_2)$ is a valid symmetric linear determinantal representation of f

Capricorn curve



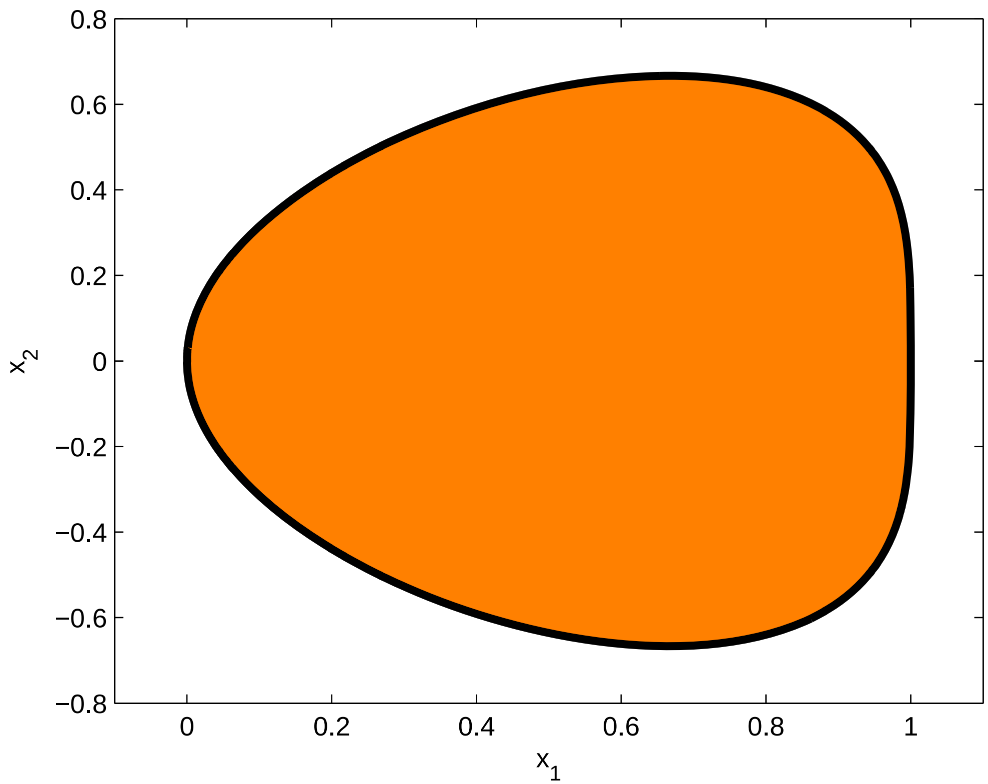
$$f(x) = x_1^2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2 - x_2)^2$$

Capricorn LMI

$$F(x) = \begin{bmatrix} 1960 - 868x - 1924y & -952 - 940x + 740y & -168 + 180x + 180y & 56 - 4x - 52y \\ -952 - 940x + 740y & 776 + 540x + 476y & -8 - 36x - 84y & -72 + 20x + 52y \\ -168 + 180x + 180y & -8 - 36x - 84y & 40 + 60x + 92y & 8 + 20x - 28y \\ 56 - 4x - 52y & -72 + 20x + 52y & 8 + 20x - 28y & 8 - 4x - 4y \end{bmatrix}$$

Definite around $e = (0, 1/2)$ hence LMI representable

Bean curve



$$f(x) = x_1^4 + x_1^2 x_2^2 + x_2^4 - x_1(x_1^2 + x_2^2)$$

Bean determinant

$$F(x) = \begin{bmatrix} x_1 & x_2 & x_1 & x_2 \\ x_2 & 1 & x_2 & 1 - x_1 \\ x_1 & x_2 & 0 & 0 \\ x_2 & 1 - x_1 & 0 & 1 - x_1 \end{bmatrix}$$

Indefinite around $e = (1/2, 0)$

Not rigidly convex hence not LMI representable

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Cubics

When deg $f(x) = 3$ the genus of $f(x)$ can be 0 (rational, or singular cubic) or 1 (elliptic, or smooth cubic)

Homogeneize $f(x_0, x_1, x_2) = x_0^3 f(\frac{x_1}{x_0}, \frac{x_2}{x_0})$, define Hessian matrix

$$H(f(x)) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right]_{i,j=0,1,2}$$

which is real symmetric linear, and Hessian

$$h(x) = \det H(f(x))$$

which is cubic

An elliptic curve has 9 flexes x^* (3 of which are real) satisfying

$$f(x^*) = h(x^*) = 0$$

Parametrized Hessian

$f(x)$ and $h(x)$ share the same flexes and we know a symmetric linear determinantal representation for $h(x)$, so use linear homotopy to find one for $f(x)$ (thanks to Frédéric Han)

For $t \in \mathbb{R}$ define parametrized Hessian

$$g(x,t) = \det H(tf(x) + h(x))$$

and find t^* satisfying

$$g(x^*,t^*) = f(x^*)$$

at a real flex x^* by solving equation of degree 3

Three distinct determinantal representations not equivalent by congruence transformation, one of which is definite hence LMI

Elliptic curve

Find a symmetric linear determinantal representation for

$$f(x) = x_1^3 - x_2^2 - x_1$$

First build Hessian

$$h(x) = \det H(f(x)) = 8(x_0^3 + 3x_0x_1^2 - 3x_1x_2^2)$$

Parametrized Hessian

$$g(t,x) = \det H(tf(x)+h(x)) = 24t^3x_0x_1^2 - 576t^2x_0^2x_1 + \cdots + 110592x_1^3$$

matches $f(x)$ at flex $x_0^* = 0$ for

$$t^* \in \{0, 24, -24\}$$

yielding the following three representations...

Elliptic curve

$$F^1(x) = \begin{bmatrix} 1 & -x_2 & x_1 \\ -x_2 & -x_1 & 0 \\ x_1 & 0 & 1 \end{bmatrix}$$

$$F^2(x) = 4^{-\frac{1}{3}} \begin{bmatrix} 1 + 3x_1 & -x_2 & -1 + x_1 \\ -x_2 & -1 - x_1 & -x_2 \\ -1 + x_1 & -x_2 & 1 - x_1 \end{bmatrix}$$

$$F^3(x) = 4^{-\frac{1}{3}} \begin{bmatrix} 1 - 3x_1 & -x_2 & 1 + x_1 \\ -x_2 & 1 - x_1 & x_2 \\ 1 + x_1 & x_2 & 1 + x_1 \end{bmatrix}$$

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Dixon's construction

A. C. Dixon described in 1902 an explicit construction of a symmetric linear determinantal representation of a plane algebraic curve based on the knowledge of a contact curve

Given a curve $f(x) = 0$ of degree d, a contact curve $g(x) = 0$ is a curve of degree $d - 1$ touching $f(x) = 0$ at $\frac{1}{2}d(d-1)$ points

Once $g(x)$ is given, a determinantal representation for $f(x)$ follows by simple linear algebra

Better illustrated with an example..

Adjoint matrix

Consider the elliptic curve

$$f(x) = 1 - 2x_1 - x_1^2 - x_2^2 + 2x_1^3 = 0$$

with (known) determinantal representation

$$F(x) = \begin{bmatrix} 1 & x_1 & 0 \\ x_1 & 1 & x_2 \\ 0 & x_2 & 1 - 2x_1 \end{bmatrix}$$

Consider its adjoint (or adjugate) matrix

$$V(x) = F^{-1}(x)f(x) = \begin{bmatrix} 1 - 2x_1 - x_2^2 & x_1(-1 + 2x_1) & x_1x_2 \\ x_1(-1 + 2x_1) & 1 - 2x_1 & -x_2 \\ x_1x_2 & -x_2 & 1 - x_1^2 \end{bmatrix}$$

which contains quadratic cofactors

Quotient ring

Since $F(x)V(x) = f(x)I$, the cofactors in a column of $V(x)$ generate a basis for the quotient ring $\mathbb{R}[x]/f(x)$

$$\begin{bmatrix} 1 & x_1 & 0 \\ x_1 & 1 & x_2 \\ 0 & x_2 & 1-2x_1 \end{bmatrix} \begin{bmatrix} 1-2x_1-x_2^2 & x_1(-1+2x_1) & x_1x_2 \\ x_1(-1+2x_1) & 1-2x_1 & -x_2 \\ x_1x_2 & -x_2 & 1-x_1^2 \end{bmatrix} = \begin{bmatrix} f(x) & 0 & 0 \\ 0 & f(x) & 0 \\ 0 & 0 & f(x) \end{bmatrix}$$

Net of contact curves

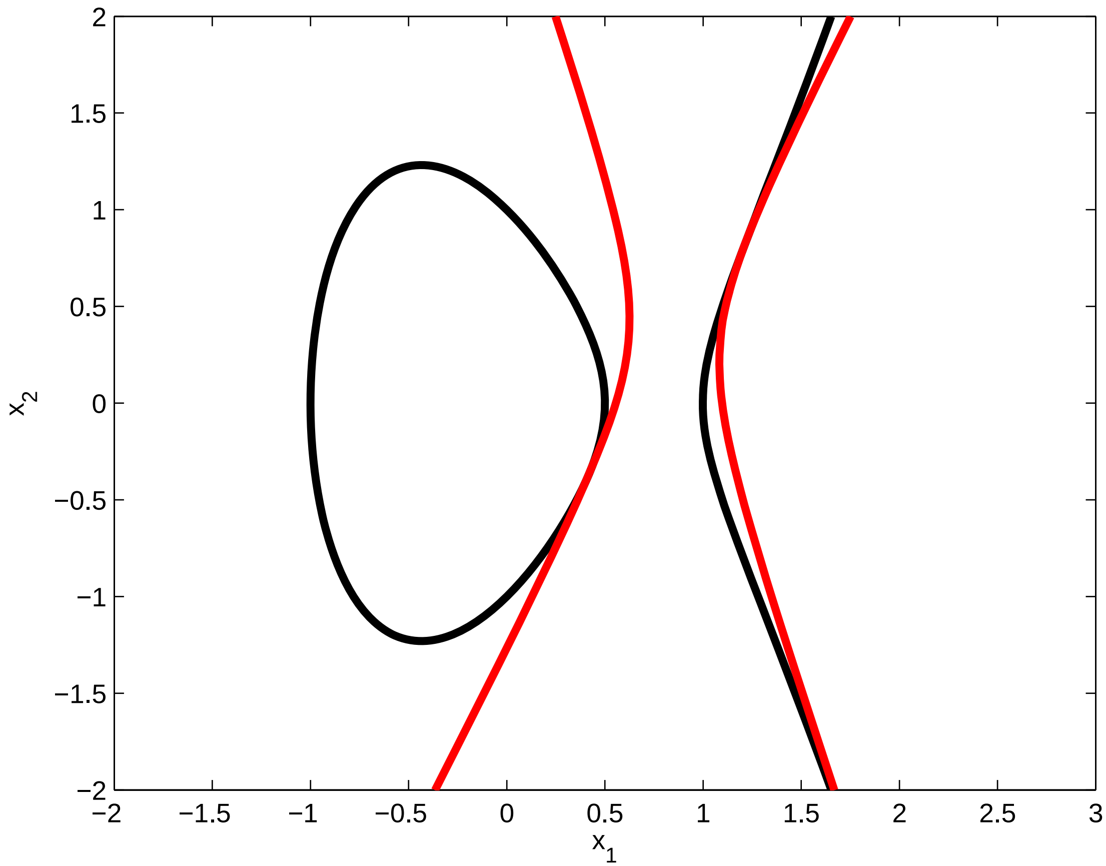
For all $v \in \mathbb{R}^3$ the conic

$$g(x, v) = v^T V(x) v = 0$$

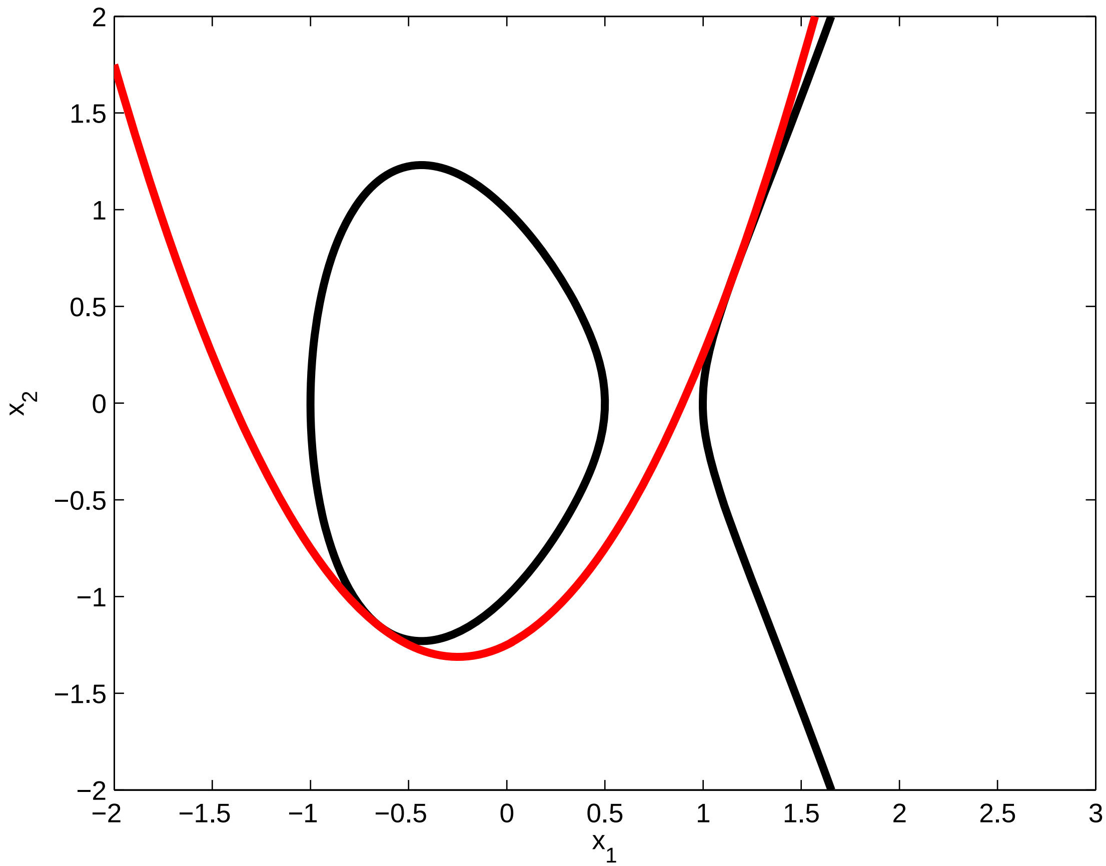
is a contact curve touching $f(x) = 0$ at 3 points

Here are some random contact conics for our cubic..

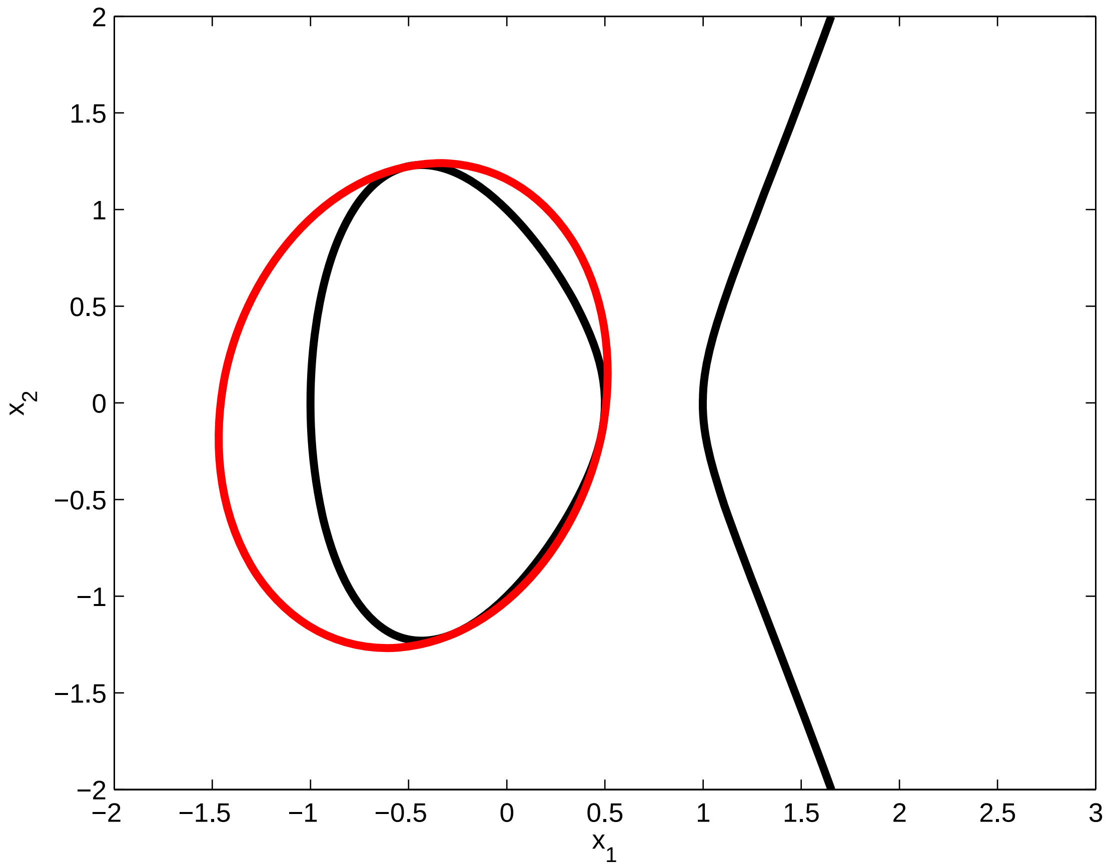
Contact curves



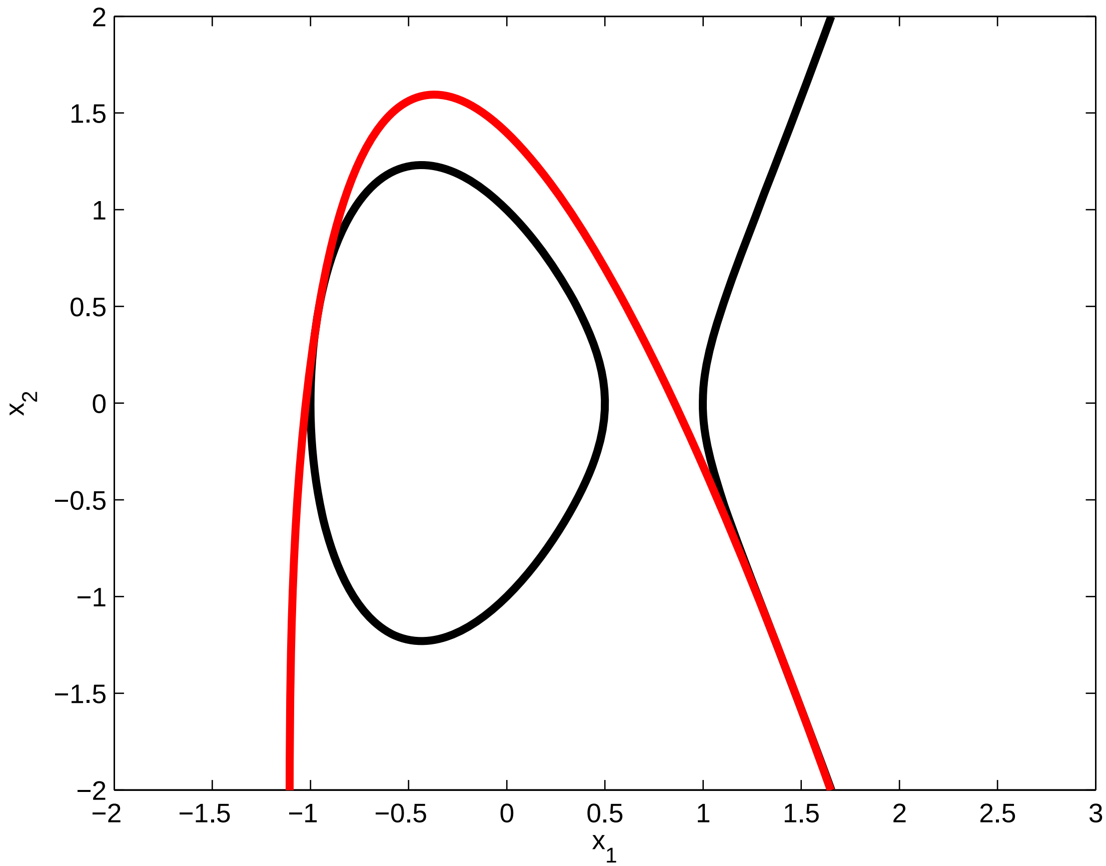
Contact curves



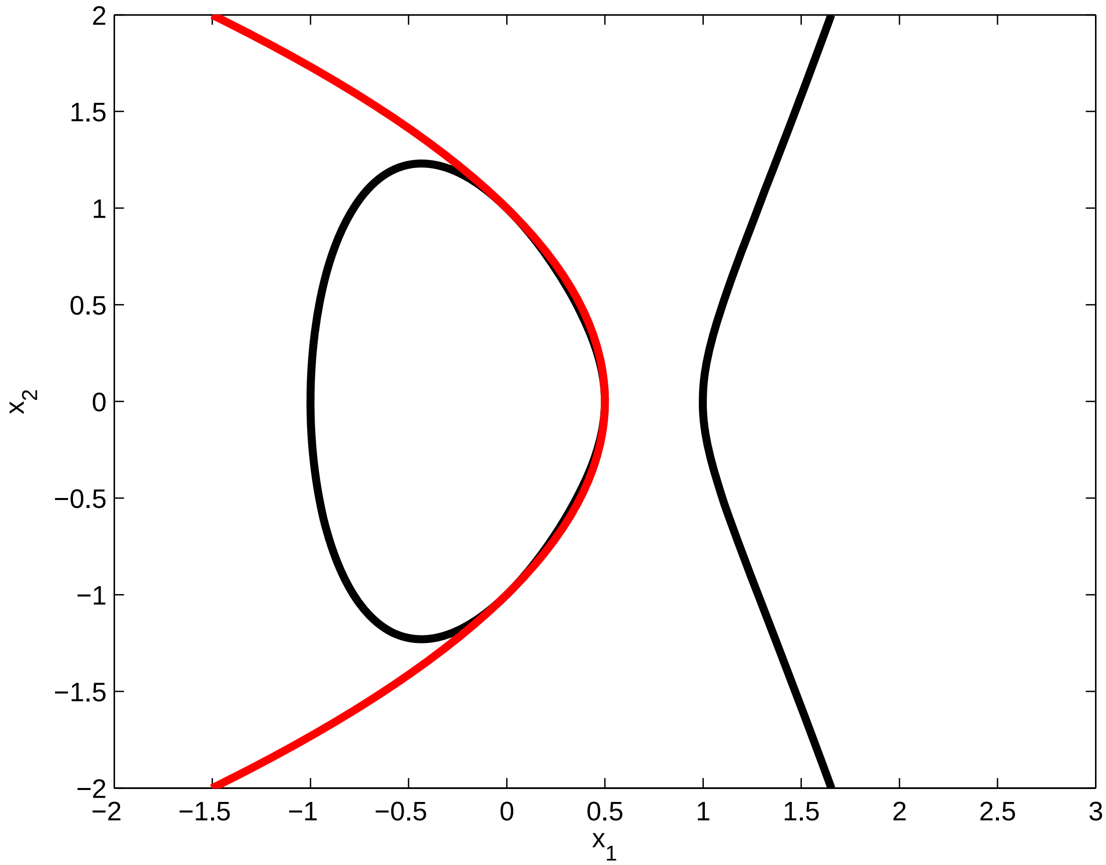
Contact curves



Contact curves



Contact parabola



Filling in the adjoint

$$V(x) = \begin{bmatrix} 1 - 2x_1 - x_2^2 & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

Let (1,1) be the contact parabola

Filling in the adjoint

$$V(x) = \begin{bmatrix} 1 - 2x_1 - x_2^2 & ? & ? \\ x_1(-1 + 2x_1) & ? & ? \\ x_1 x_2 & ? & ? \end{bmatrix}$$

Build basis for quotient ring $\mathbb{R}[x]/f(x)$

Filling in the adjoint

$$V(x) = \begin{bmatrix} 1 - 2x_1 - x_2^2 & x_1(-1 + 2x_1) & x_1 x_2 \\ x_1(-1 + 2x_1) & ? & ? \\ x_1 x_2 & ? & ? \end{bmatrix}$$

All these contact conics share common touching points

Filling in the adjoint

$$V(x) = \begin{bmatrix} 1-2x_1-x_2^2 & x_1(-1+2x_1) & x_1x_2 \\ x_1(-1+2x_1) & \color{red}{1-2x_1} & ? \\ x_1x_2 & ? & ? \end{bmatrix}$$

(2,2) shares contact points with (1,1) and (2,1)

Filling in the adjoint

$$V(x) = \begin{bmatrix} 1 - 2x_1 - x_2^2 & x_1(-1 + 2x_1) & x_1 x_2 \\ x_1(-1 + 2x_1) & 1 - 2x_1 & ? \\ x_1 x_2 & ? & 1 - x_1^2 \end{bmatrix}$$

(3,3) shares contact points with (1,1) and (3,1)

Filling in the adjoint

$$V(x) = \begin{bmatrix} 1 - 2x_1 - x_2^2 & x_1(-1 + 2x_1) & x_1 x_2 \\ x_1(-1 + 2x_1) & 1 - 2x_1 & -x_2 \\ x_1 x_2 & -x_2 & 1 - x_1^2 \end{bmatrix}$$

(3,2) shares contact points with (2,2) and (3,3)

Inverting the adjoint

$$f(x)V^{-1}(x) = \begin{bmatrix} 1 & x_1 & 0 \\ x_1 & 1 & x_2 \\ 0 & x_2 & 1 - 2x_1 \end{bmatrix}$$

Invert adjoint matrix to recover linear representation

Dixon's construction

Given $f(x)$ of degree d, and a contact curve $g(x)$ of degree $d-1$:

- let $g(x)$ be a diagonal entry in $V(x)$
- derive the whole row and column in $V(x)$ by generating a basis of the quotient ring $\mathbb{R}[x]/f(x)$
- derive the remaining entries in $V(x)$ by solving linear equations in the quotient ring
- compute $F(x) = f^{d-2}(x)V(x)^{-1}$

A key issue remains: how can we algebraically and systematically find a contact curve $g(x)$?

Impact of the choice of the contact points ?

Definiteness of the representation ?

Open problems

For genus zero curves:

- how many distinct parametrizations ?
- how can we detect/enforce definiteness ?

For positive genus curves:

- how can we find a contact curve ?
- how can we detect/enforce definiteness ?
- for quartics, should we use the bitangents (Edge 1935) ?
- can we singularize the curve (inverse quadratic mappings of Hilbert/Hurwitz to solve singularities) and use the Bezoutian ?

Example of 3D LMI set

$$\mathcal{F} = \{x \in \mathbb{R}^3 \; : \; \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{bmatrix} \succeq 0\}$$