# The multi-modal traveling salesman problem

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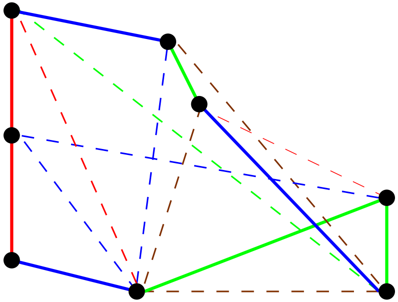


Data:

$G = (V, E)$ : an undirected valuated graph
$C$ is a set of colors
Each $e \in E$ has a color $k \in C$

Goal:

Find a Hamiltonian cycle
Two objectives:

1. Minimize the total length of the cycle
2. Minimize the number of colors appearing on the cycle

## Integer program

Variables

$$x_e = \begin{cases} 1 & \text{if } e \in E \text{ is used,} \\ 0 & \text{otherwise.} \end{cases}$$

$$u_k = \begin{cases} 1 & \text{if } k \in C \text{ is used,} \\ 0 & \text{otherwise.} \end{cases}$$

Constants and notations

$$\forall e \in E,\ \delta(e) = k \in C \text{ the color of } e$$

$$\forall k \in C,\ \zeta(k) = \{e \in E | \delta(e) = k\}$$

$$\forall S \subset V,\ \omega(S) = \{e = (i,j) \in E | i \in S \text{ and } j \in V \setminus S\}$$

## Integer program

Objective functions

$$\min \sum_{e \in E} c_e x_e$$

$$\min \sum_{k \in C} u_k$$

Constraints

$$\begin{aligned}
\sum_{e \in \omega(\{i\})} x_e &= 2 && \forall i \in V \\
\sum_{e \in \omega(S)} x_e &\geq 2 && \forall S \subset V, 3 \leq |S| \leq |V| - 3 \\
x_e &\leq u_{\delta(e)} && \forall e \in E \\
x_e &\in \{0, 1\} && \forall e \in E \\
u_k &\in \{0, 1\} && \forall k \in C
\end{aligned}$$

# Valid constraints

$$
\begin{aligned}
u_k &\leq \sum_{e \in \zeta(k)} x_e && \forall k \in C \\
\sum_{k \in C} \gamma_i^k u_k &\geq 2 && \forall i \in V \\
\sum_{k \in C} \lambda_k(S) u_k &\geq 2 && \forall S \in V, 3 \leq |S| \leq |V| - 3
\end{aligned}
$$

with

$$
\gamma_i^k = \begin{cases} 0 & \text{if } \nexists e \in \omega(\{i\}), e \in \zeta(k), \\ 1 & \text{if } \exists! e \in \omega(\{i\}), e \in \zeta(k), \\ 2 & \text{otherwise.} \end{cases} \qquad \lambda_k(S) = \begin{cases} 0 & \text{if } \nexists e \in \omega(S), e \in \zeta(k), \\ 1 & \text{if } \exists! e \in \omega(S), e \in \zeta(k), \\ 2 & \text{otherwise.} \end{cases}
$$

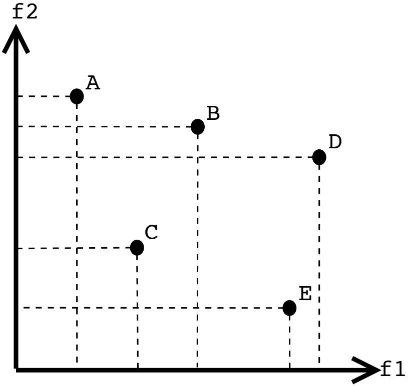
# Multi-objective optimization problem

$$(PMO) = \begin{cases} min\ F(x) = (f_1(x), f_2(x), \dots, f_n(x)) \\ s.t.\ x \in \Omega \end{cases}$$

with:

- $n \geq 2$ : number of objectives
- $F = (f_1, f_2, \dots, f_n)$ : vector of functions to optimize
- $\Omega \subseteq \mathbb{R}^m$ : set of feasible solutions
- $x = (x_1, x_2, \dots, x_m) \in \Omega$ : a feasible solution
- $\mathcal{Y} = F(\Omega)$ : objective space
- $y = (y_1, y_2, \dots, y_n) \in \mathcal{Y}$ avec $y_i = f_i(x)$ : a point in the objective space

A solution $x$ dominates ($\preceq$) a solution $y$ if and only if
$\forall i \in \{1, \ldots, n\}, f_i(x) \leq f_i(y)$ and $\exists i \in \{1, \ldots, n\}$ such that $f_i(x) < f_i(y)$.



# Exact algorithms for MOP

| | $n = 2$ | $n \geq 2$ |
|---|---|---|
| Iteration | Two-Phase method<br>PPM, $\epsilon$-constraint method | K-PPM |
| Multi-objective method | | [Sourd, Spanjaard, 2008] (*) |

(*) does not work if the aggregated problem is NP-hard

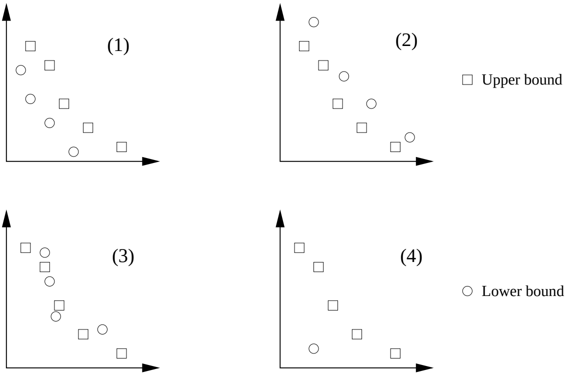
$\Rightarrow$ a multi-objective branch-and-cut algorithm for multi-objective integer programs

Lower bound $\Leftrightarrow$ multi-objective linear program

Possibility to use scalar techniques to solve it to optimality (or a subset that can be extended)

Upper bound = set of non-dominated solutions found during the search

Lower bound = set of non-dominated points in the objective space such that all feasible solutions are dominated by these points



# A multi-objective branch-and-cut algorithm

STEP 1 (Root of the tree)
Generate an initial upper bound $ub$
Define a first sub-problem
Insert the sub-problem in a list $L$

STEP 2 (Stopping criterion)
If $L = \emptyset$ then STOP, else choose a sub-problem from $L$ and remove it from $L$

STEP 3 (Sub-problem solution)
Solve the sub-problem to obtain the lower bound $lb$

STEP 4 (Constraint generation)
Try to insert integer solutions from $lb$ in $ub$
**if** $lb = \emptyset$ or $ub \preceq lb$ **then**
  Go to STEP 2.
**else if** violated constraints are identified for $\{x \in lb | \nexists y \in ub, y \preceq x\}$ **then**
  Add them to the model and go to STEP 3.
**else**
  Go to STEP 5.
**end if**

STEP 5 (Branching)
Branch on variable and introduce new sub-problems in $L$. Go to STEP 2.

## Computation of the lower bound

Initial sub-problem :

$$\min \quad \sum_{e\in E} c_e x_e$$

$$\min \quad \sum_{k\in C} u_k$$

$$\begin{aligned}
\sum_{e\in\omega(\{i\})} x_e &= 2 && \forall i \in V\\
x_e &\leq u_{\delta(e)} && \forall e \in E\\
u_k &\leq \sum_{e\in\zeta(k)} x_e && \forall k \in C\\
\sum_{k\in C} \gamma_i^k u_k &\geq 2 && \forall i \in V\\
0 \leq x_e &\leq 1 && \forall e \in E\\
0 \leq u_k &\leq 1 && \forall k \in C
\end{aligned}$$

# Computation of the lower bound

Solve the following problem for different values of $\epsilon$

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e + m \sum_{k \in C} u_k \\
& \sum_{e \in \omega(\{i\})} x_e = 2 && \forall i \in V \\
& x_e \leq u_{\delta(e)} && \forall e \in E \\
& u_k \leq \sum_{e \in \zeta(k)} x_e && \forall k \in C \\
& \sum_{k \in C} \gamma_i^k u_k \geq 2 && \forall i \in V \\
& \sum_{k \in C} u_k \leq \epsilon \\
& 0 \leq x_e \leq 1 && \forall e \in E \\
& 0 \leq u_k \leq 1 && \forall k \in C
\end{aligned}
$$

After founding non-dominated solution for a given $\epsilon$, identify violated constraints and add them

# Computation of the lower bound

*ub* is the upper bound. Set $L_{\text{tabu}} \leftarrow \emptyset$ and *continue* $\leftarrow$ TRUE

**while** *continue* is TRUE **do**
  *continue* $\leftarrow$ FALSE
  *pruned* $\leftarrow$ TRUE
  Set $\epsilon \leftarrow \alpha$ with $\alpha$ an integer such that $\alpha \notin L_{\text{tabu}}$ and $\nexists \beta \notin L_{\text{tabu}}$ such that $\alpha < \beta \leq |C|$.

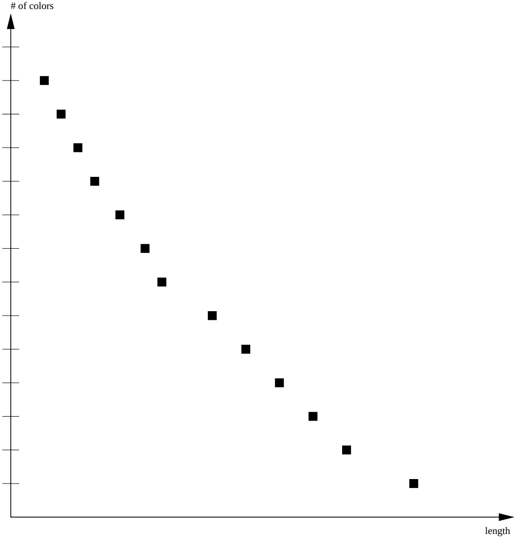
  **while** $\epsilon \neq 0$ **do**
    Solve the linear program. Let $(x^*, u^*)$ be the optimal solution and $l^*$ the length of the solution and $o^*$ the number of colors used.

    **if** a solution is found **then**
      **if** the solution is feasible and integer or the solution is dominated by *ub* **then**
        **if** the solution is feasible and integer **then**
          Try to add it in *ub* and update *ub* if necessary
        **end if**
        $L_{\text{tabu}} \leftarrow \{\lceil o^* \rceil \dots \epsilon\}$
      **else**
        *pruned* $\leftarrow$ FALSE
        **if** constraints violated by $(x^*, u^*)$ are identified **then**
          Stock them
        **end if**
      **end if**
    **else**
      $L_{\text{tabu}} \leftarrow \{1 \dots \epsilon\}$
    **end if**

    Set $\epsilon \leftarrow \alpha$ with $\alpha$ an integer such that $\alpha \notin L_{\text{tabu}}$ and $\nexists \beta \notin L_{\text{tabu}}$ such that $\alpha < \beta < o^*$.
  **end while**

  **if** violated constraints have been found **then**
    Add them to the model
    *continue* $\leftarrow$ TRUE
  **end if**
**end while**

# Computation of the lower bound



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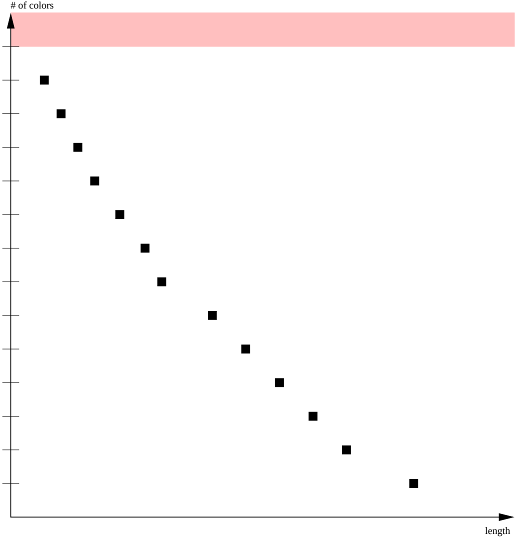

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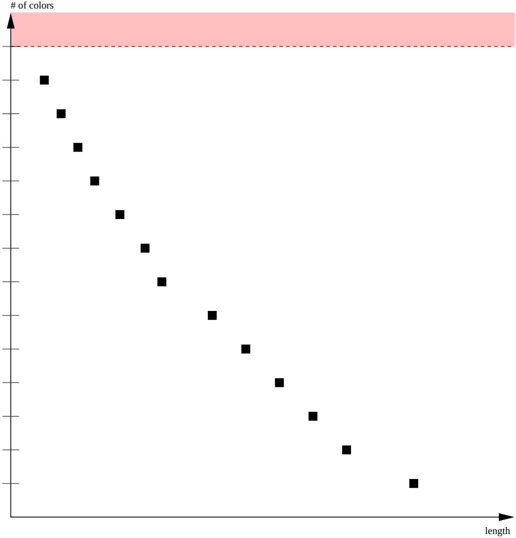

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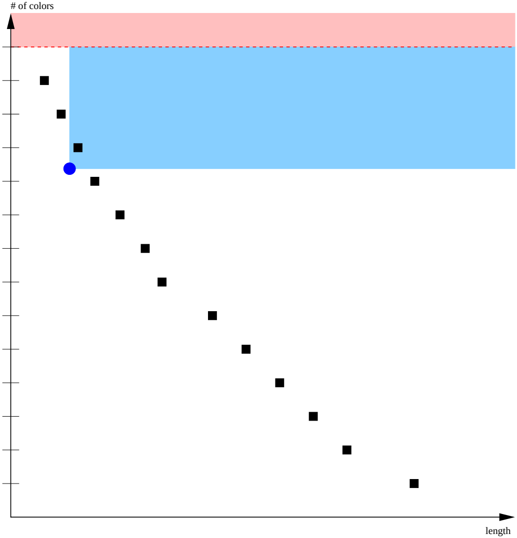

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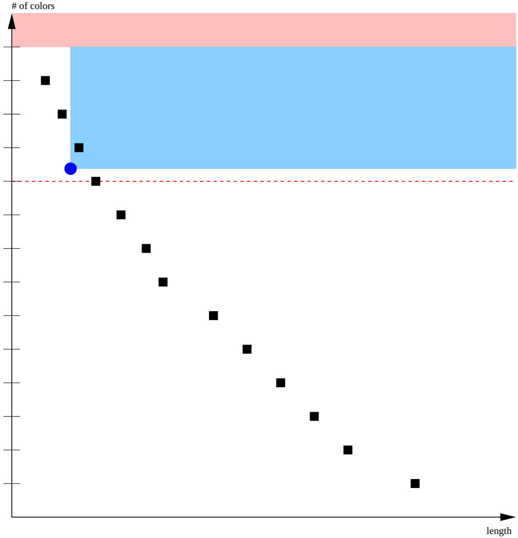

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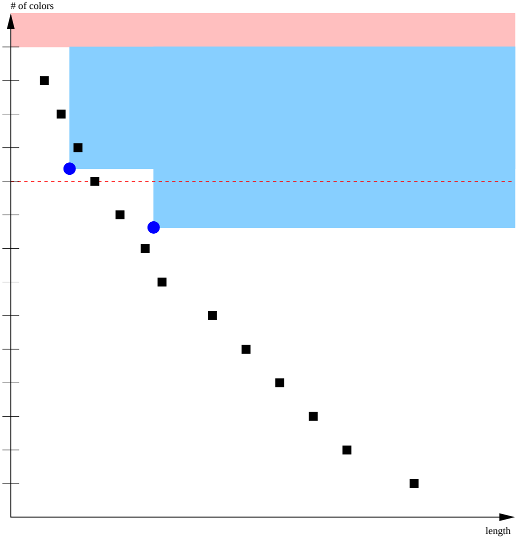

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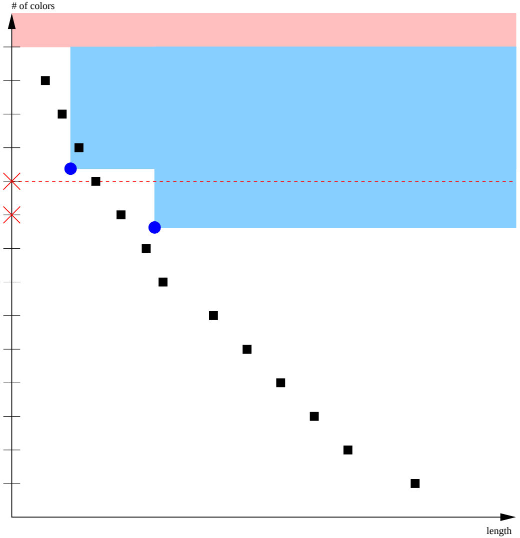

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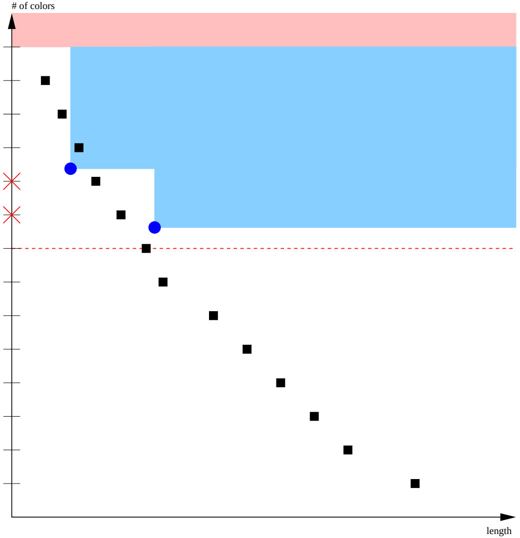

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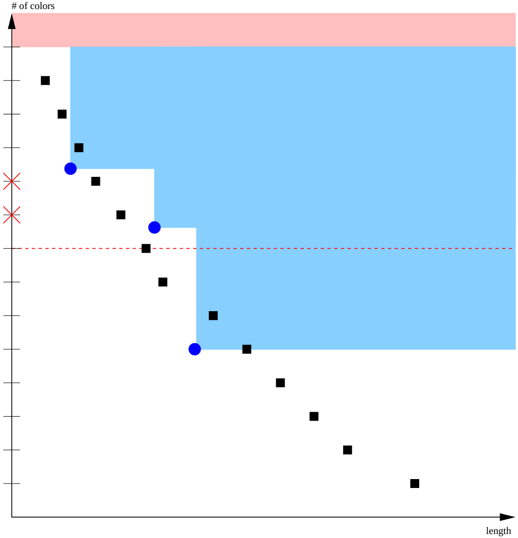

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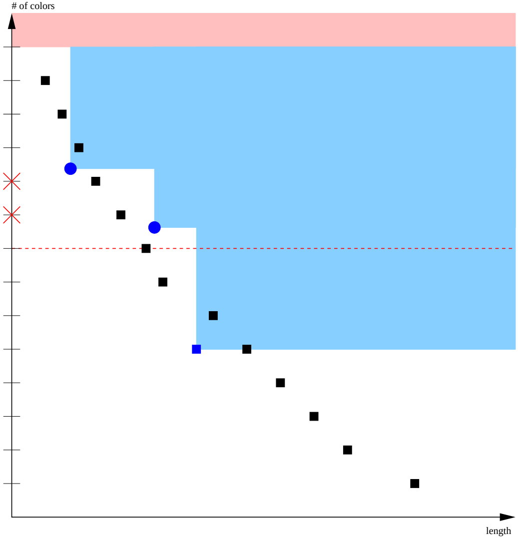

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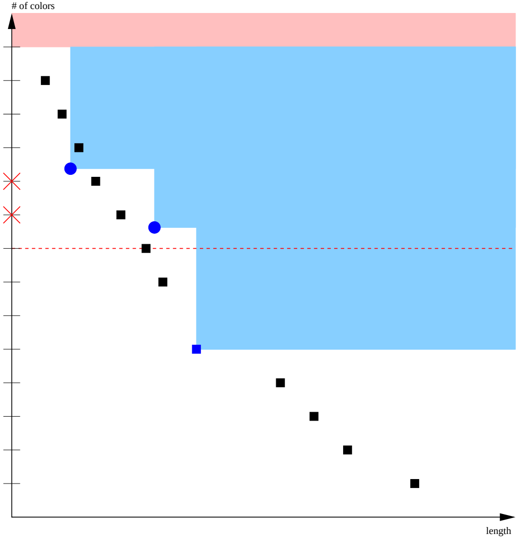

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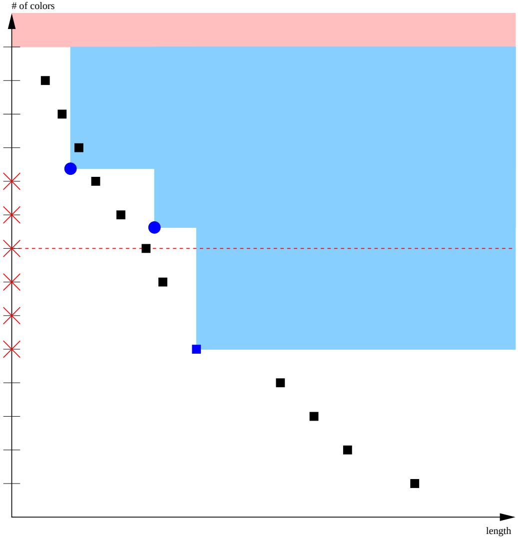

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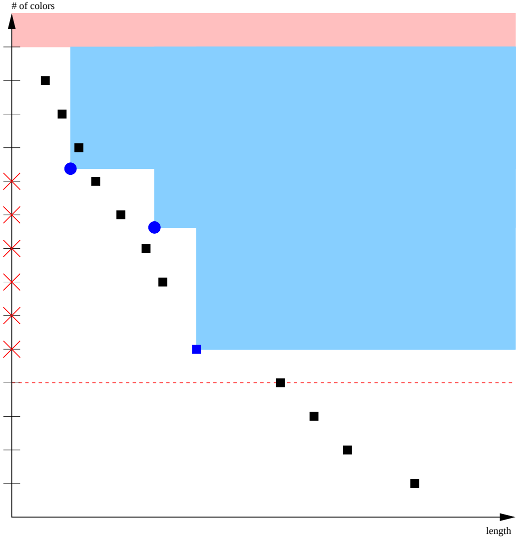
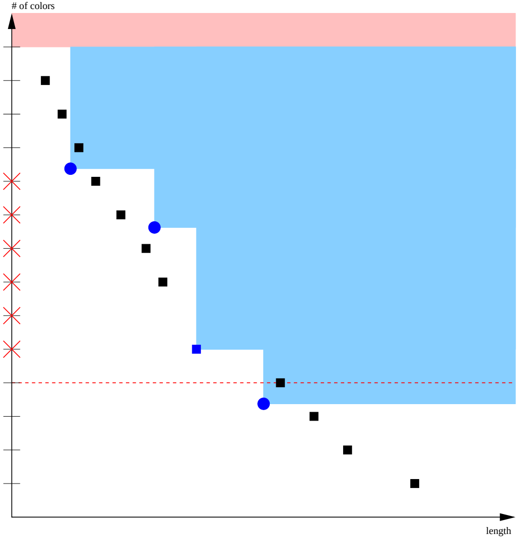
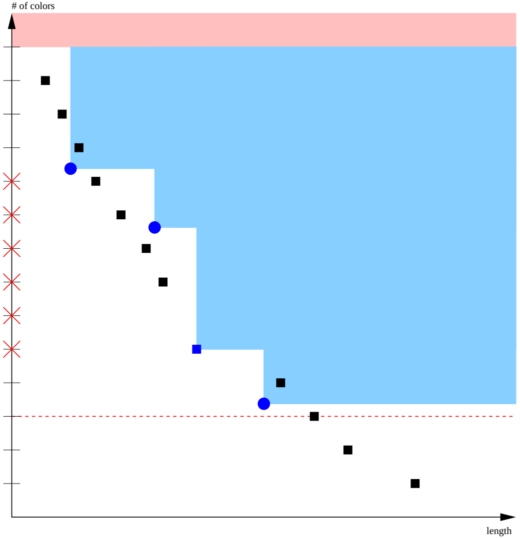




# of colors
length

# Computation of the lower bound



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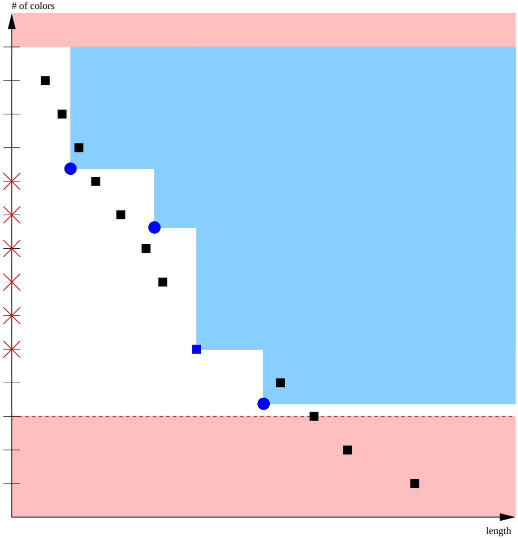

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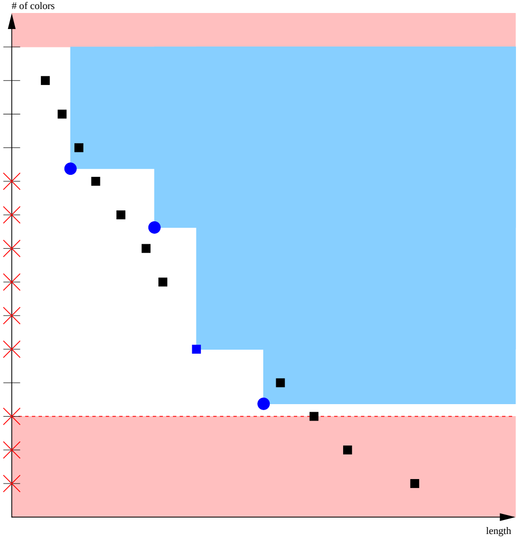

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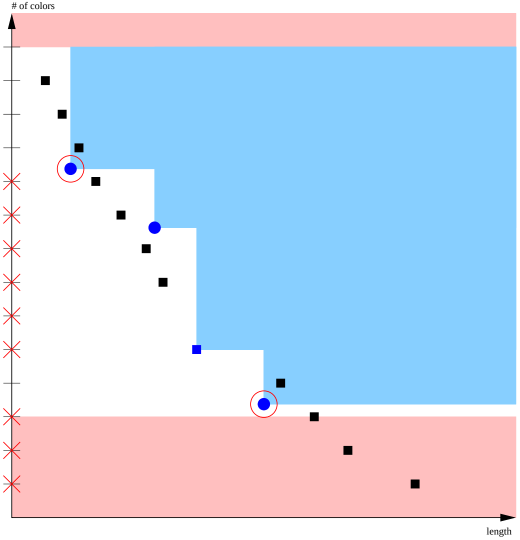

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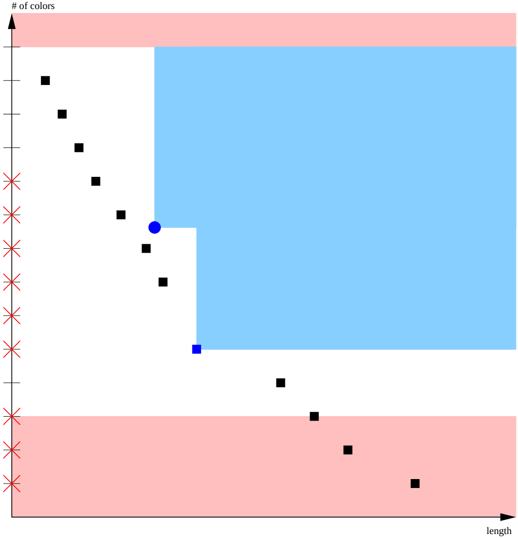

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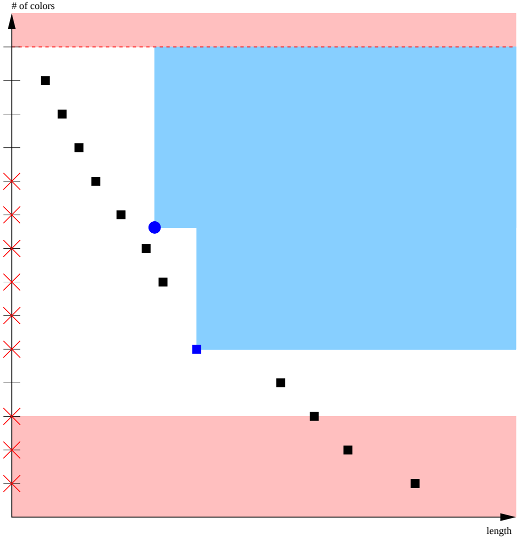

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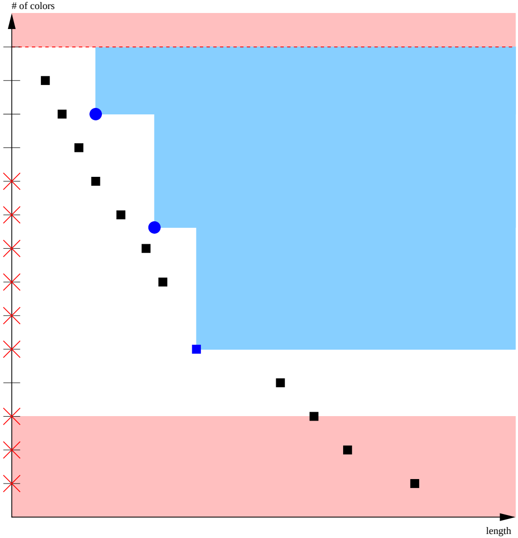

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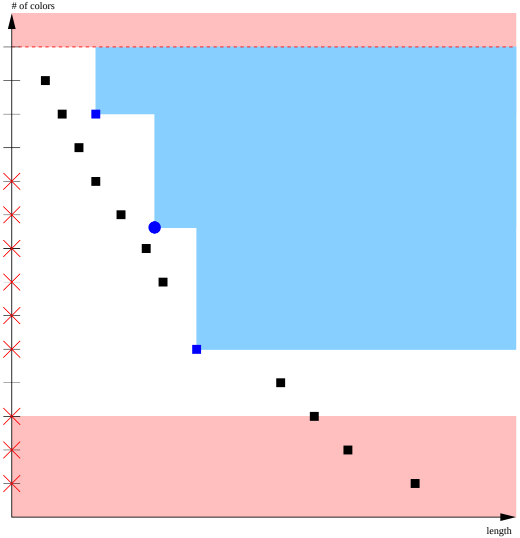

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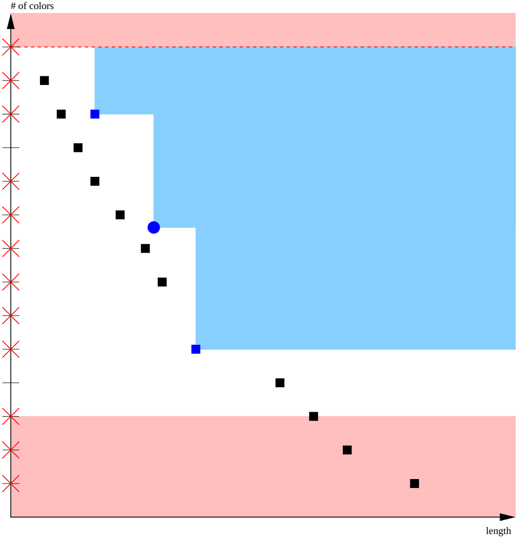

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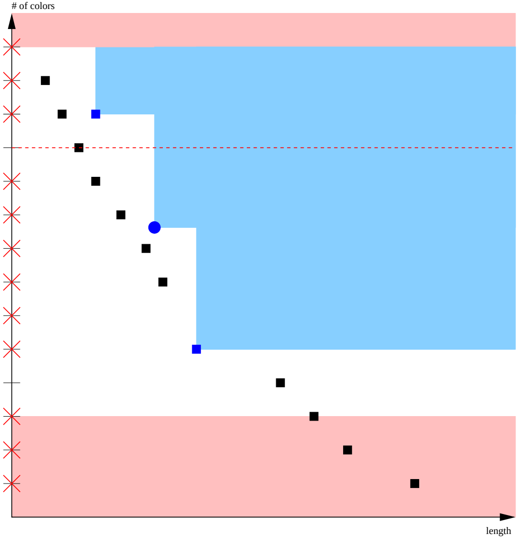

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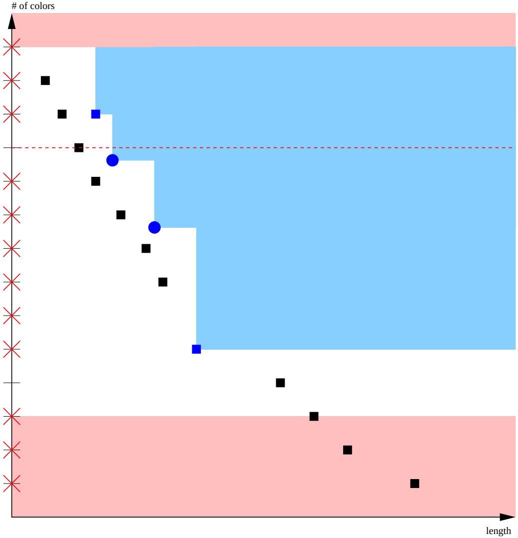
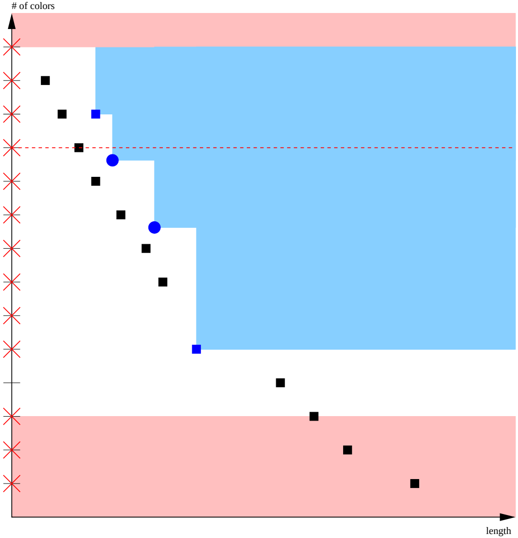
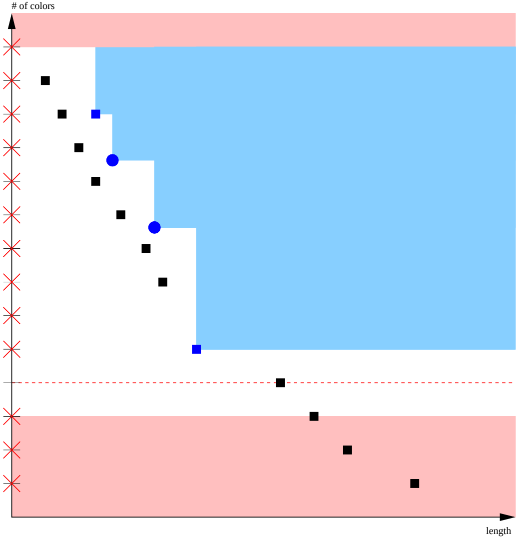


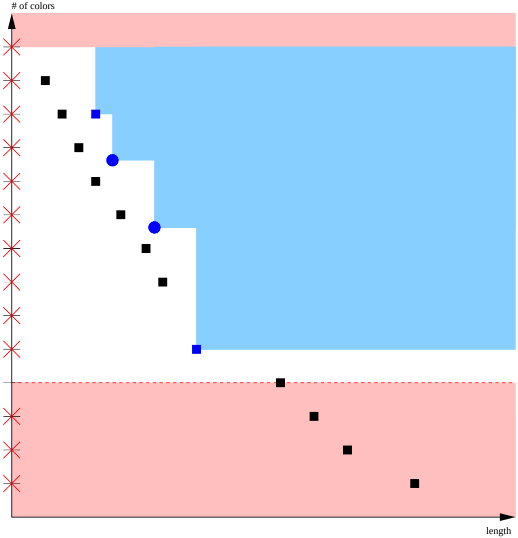
## Computation of the lower bound





# of colors
length

# Computation of the lower bound



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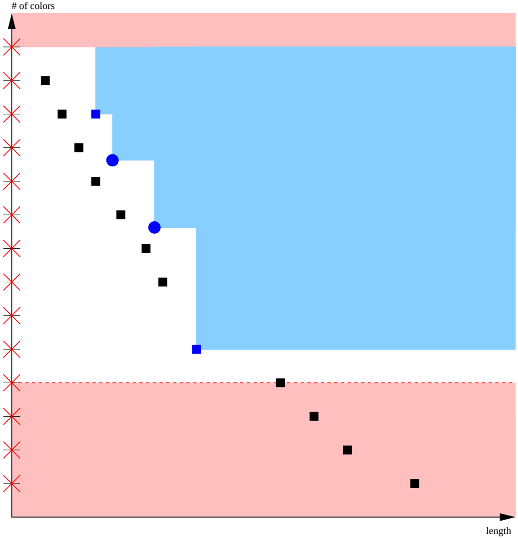


# Constraint generation, cutting, and branching

## Constraint generation

Connectivity constraints → min-cut problem

Call to a CONCORDE function [Padberg & Rinaldi, 1990]

## Cutting

$\forall \epsilon$, the sub-problem is infeasible

$\forall \epsilon$, the solution is feasible $ub$

$\forall \epsilon$, the solution is dominated by $ub$

$\exists \epsilon$, the solution is not dominated by $ub$ and no constraint can be added

## Branching

First on the $u_k$ variables then on the $x_e$

Priority on the variable that is non integral for the most values of $\epsilon$

# Computation of the initial upper bound

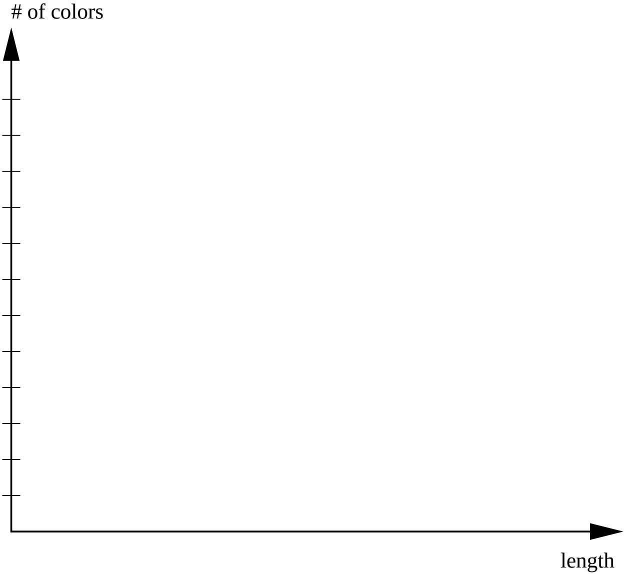
Repeat the following sequence for different values of $\epsilon$

STEP 1: Solve the following mixed integer program :

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e + m \sum_{k \in C} u_k \\
\sum_{e \in \omega(\{i\})} x_e &= 2 && \forall i \in V \\
x_e &\leq u_{\delta(e)} && \forall e \in E \\
u_k &\leq \sum_{e \in \zeta(k)} x_e && \forall k \in C \\
\sum_{k \in C} u_k &\leq \epsilon \\
0 \leq x_e &\leq 1 && \forall e \in E \\
u_k &\in \{0,1\} && \forall k \in C
\end{aligned}
$$

STEP 2: Solve a TSP on $G' = (V, E')$ with $E' = \{e \in E | u_{\delta(e)} = 1\}$

# Computation of the initial upper bound



# Computation of the initial upper bound

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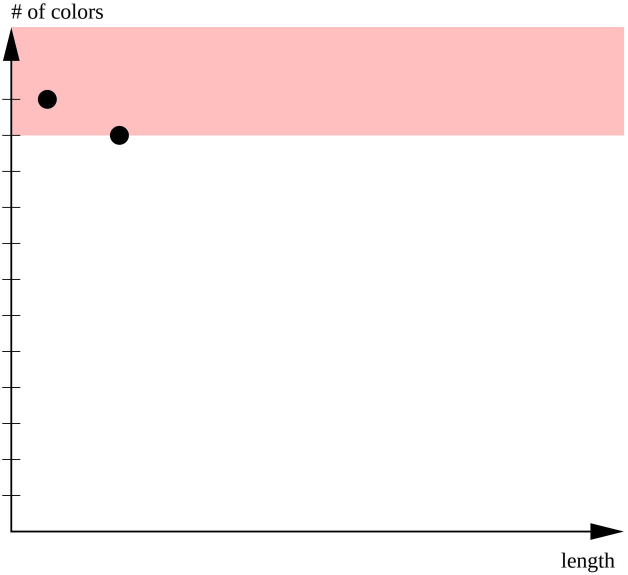
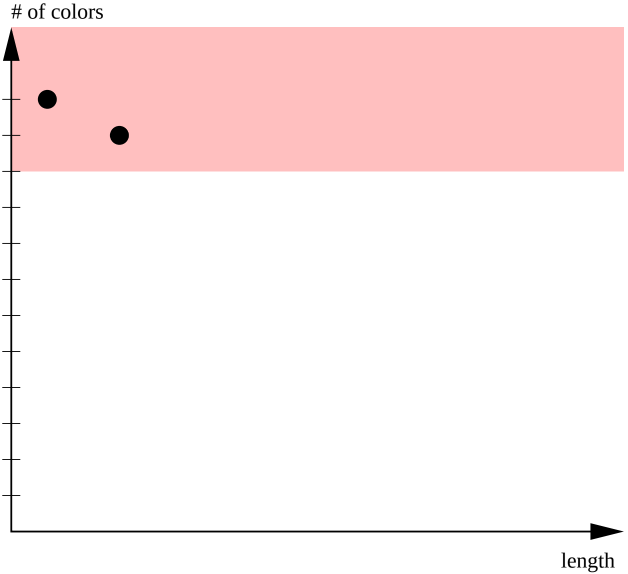


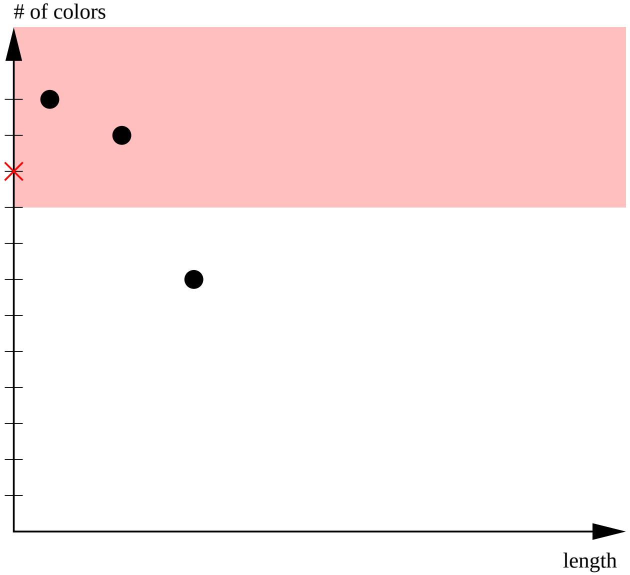


# of colors
length


# of colors
length

# of colors
length

# Computation of the initial upper bound



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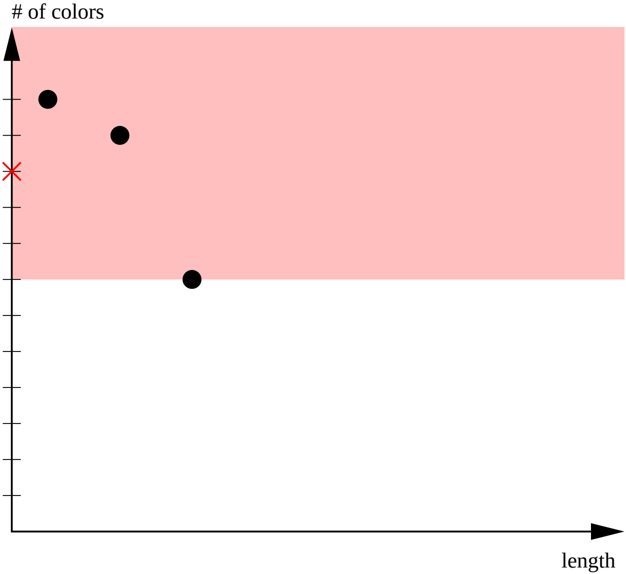
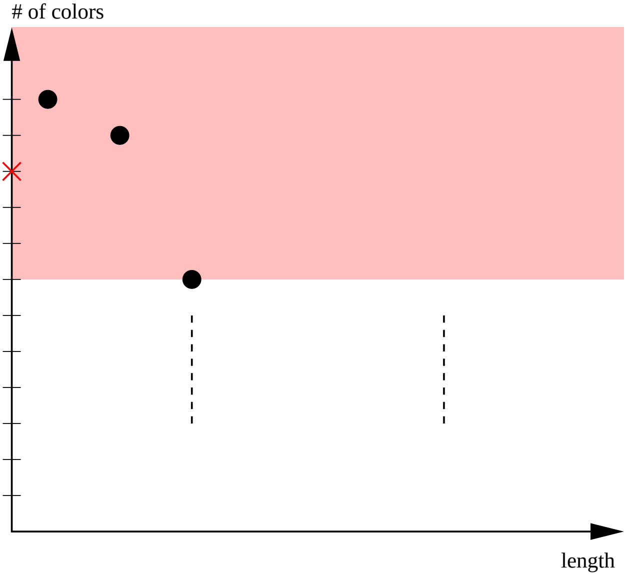
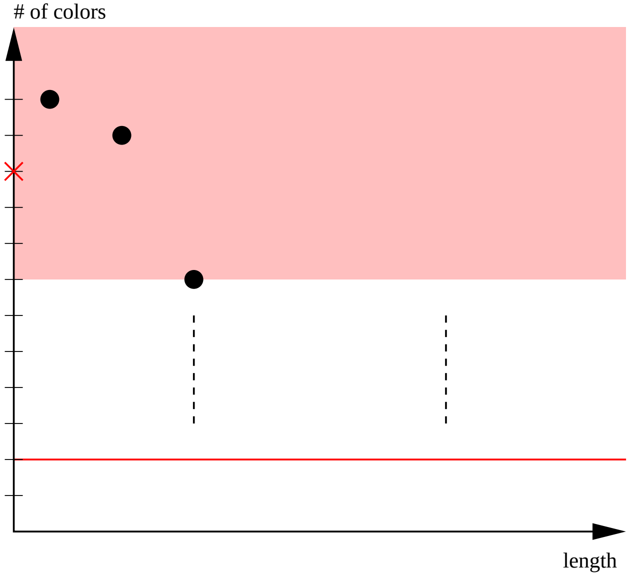



# of colors
length



# of colors
length

# Computational results

| $\|C\|$ | $\|V\|$ | #nodes | #u | #x | #cut | #Pareto | #Parub | #time |
|---|---|---|---|---|---|---|---|---|
| 20 | 20 | 270.1 | 129.0 | 5.6 | 57.4 | 10.3 | 4.4 | 4.1 |
| 20 | 30 | 573.6 | 256.2 | 30.1 | 132.8 | 13.9 | 4.8 | 28.2 |
| 20 | 40 | 1002.9 | 410.2 | 90.8 | 225.6 | 15.9 | 5.3 | 100.8 |
| 20 | 50 | 1738.6 | 563.7 | 305.1 | 366.7 | 17.4 | 5.9 | 347.9 |
| 30 | 20 | 506.1 | 248.2 | 4.4 | 85.0 | 12.4 | 4.5 | 10.8 |
| 30 | 30 | 1284.6 | 592.5 | 49.3 | 193.4 | 16.4 | 4.5 | 96.7 |
| 30 | 40 | 2536.7 | 1084.2 | 183.6 | 352.3 | 18.8 | 4.2 | 484.3 |
| 30 | 50 | 5519.5 | 1880.9 | 878.4 | 590.8 | 21.7 | 5.0 | 1618.9 |
| 40 | 20 | 799.6 | 396.5 | 2.8 | 100.6 | 12.1 | 4.1 | 20.6 |
| 40 | 30 | 2241.6 | 1097.4 | 22.9 | 258.8 | 17.8 | 3.6 | 247.6 |
| 40 | 40 | 5684.9 | 2606.2 | 235.8 | 535.8 | 21.7 | 3.6 | 1852.6 |
| 40 | 50 | 14297.6 | 5321.8 | 1826.5 | 870.6 | 26.6 | 5.4 | 7200.4 |
| 50 | 20 | 804.0 | 400.4 | 1.1 | 98.8 | 12.4 | 4.6 | 21.4 |
| 50 | 30 | 3682.4 | 1806.6 | 34.1 | 336.4 | 18.8 | 3.7 | 541.4 |
| 50 | 40 | 10451.6 | 5053.2 | 172.1 | 748.6 | 23.9 | 4.2 | 5865.0 |
| 50 | 50 | 18975.5 | 8284.5 | 1202.8 | 1156.4 | 27.7 | 4.2 | 19942.8 |

# Conclusions and perspectives

- Bi-objective variant of the TSP
- (Multi-objective) branch-and-cut algorithm

Further works:

- Identify new valid constraints → variables $u_k$
- Minimization of the number of color swapping